Tissot Indicatrix: A Means To Map Distortion Analysis

Bishnu Bahadur Karkey
Project chief, National Land Use Project

Tissot's Indicatrix

In 1859, N.A. Tissot published a classic analysis of the distortion which occurs on map projection. Tissot showed the relationship graphically with an ellipse of distortion called an indicatrix. An infinitely small circle on the earth projects as an infinitely small ellipse on any map projection. If the projection is conformal, the ellipse is a circle, an ellipse of zero eccentricity. The ellipse has a major axis and a minor axis which are directly related to the scale distortion and to the maximum angular deformation. The values for a and b, the two axes of an ellipse on projection can be computed using the following expression:

$$a = \frac{1}{2}\left(\sqrt{m^2 + 2mn\cos\varepsilon + n^2} + \sqrt{m^2 - 2mn\cos\varepsilon + n^2}\right)$$

$$b = \frac{1}{2}\left(\sqrt{m^2 + 2mn\cos\varepsilon + n^2} - \sqrt{m^2 - 2mn\cos\varepsilon + n^2}\right)$$

Here,

m = scale along meridian at a point
n = scale along parallel at a point
ε = deviation from right angle of the intersection (parallel and meridian) $(i - 90^o = \varepsilon)$

Computation of m, n, e
Gauss coefficients

$$e = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2,$$

$$f = \frac{dx}{d\phi} \times \frac{dx}{d\lambda} + \frac{dy}{d\phi} \times \frac{dy}{d\lambda},$$

$$g = \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2$$

Distance on projection plane

$$ds' = (ed\phi^2 + 2fd\phi d\lambda + gd\lambda^2)^{\frac{1}{2}}$$

Along the meridian (at constant l) and parallel (at constant Ø) dimension of an infinitesimal segment may be computed as follows:

$$ds'_m = \sqrt{e}d\phi,\ ds'_p = \sqrt{g}d\lambda$$

Angle i between the meridian and parallel on the projection computed as follows:

$i = tan^{-1}(h/f) = cos^{-1}(f/\sqrt{(eg)}) = sin^{-1}(h/\sqrt{(ge)})$
where $h = \sqrt{(eg - f^2)} = x_\lambda y_\varphi - x_\varphi y_\lambda$
$\varepsilon = tan^{-1}(-f/h)$
where $\varepsilon = i - 90^o$, the deviation of i from a right angle on the projection.
When $\alpha = 0$
$m = \mu_{\alpha,0} = \sqrt{e/M}$ (scale at meridian)
When $\alpha = 90$
$n = \mu_{\alpha,90} = \sqrt{g/r}$ (scale at parallel)

Convergence along meridians and parallels.

$\gamma_m = tan^{-1}(x_\varphi/y_\varphi),\ \gamma_p = (x_\lambda/y_\lambda)$
angle i is considered to be clockwise from north, and its quadrant is determined by the sign of f.
If $f > 0$, i is in first quadrant
If $f < 0$, i is in second quadrant
If $f = 0$, $i = 90^o$ (parallels and meridians are orthogonal)

Required condition for

a) conformal projection

i. $\beta \approx \alpha$
ii. $f = 0$ and $(e/h) * (r/M) = 1$
iii. $m = n$ and $\in = 0$
iv. $\sqrt{e/M} = \sqrt{g/r}$ and $f = 0$

Cauchy - Riemann conditions
$(1-e^2 \sin^2\phi)\, dx/(1-e^2)d\phi = -dy/(\cos\phi\, d\lambda)$

b) Equal area projection

$$s = \int_{\phi_0}^{\phi_2}\int_{\lambda_1}^{\lambda_1+1} Mr\,d\phi\,d\lambda \text{ (Ellipsoidal)}$$

$$A = \int_{\phi_0}^{\phi_2}\int_{\lambda_1}^{\lambda_1+1} h\,d\phi\,d\lambda \text{ (Projection Plane)}$$

(i) $s \approx A$

(ii) $h = Mr$

Opening it in series,

$s = b^2 (\sin\phi + (2/3) e^2 \sin^3\phi + (3/5) e^4 \sin^5\phi + 4/7\, e^6 \sin^7\phi)$

(area from equator to ϕ, and width of 1 radian)

$$\frac{dx}{d\lambda}\frac{dy}{d\phi} - \frac{dx}{d\phi}\frac{dy}{d\lambda} = Mr$$

$$\frac{dx}{d\lambda}\frac{dy}{d\phi} - \frac{dx}{d\phi}\frac{dy}{d\lambda} = R^2 \cos\phi$$

For small scale mapping an ellipse is taken as sphere, then,

$mn \sin i - mn \cos \varepsilon = 1$

$ab=1$

For ellipsoid

$$\frac{dx}{d\lambda}\frac{dy}{d\phi} - \frac{dx}{d\phi}\frac{dy}{d\lambda} = (a^2(1-e^2) \cos\phi)/(1-e^2 \sin^2\phi)^2$$

c) Equidistant projection

If $m = 1$ then $x^2_\varphi + y^2_\varphi = M^2$

If $n = 1$ then $x^2_\lambda + y^2_\lambda = r^2$

Azimuth β of linear elements ds' on the projection

$\tan\beta = (m\, h \tan\alpha) / (er + m f \tan\alpha)$

$\tan\alpha = (n \sin i - m)/(n \cos i)$

$M = a (1-e^2)/(1-e^2 \sin^2\phi)^{3/2}$

$r = a \cos\phi / (1-e^2\sin^2\phi)^{1/2}$

α = ellipsoidal azimuth

β = projection plane azimuth

e, f are gauss coefficients

m,n are scale at a point along meridian and parallel

a is the semi major axis of the earth ellipsoid.

Linear scale

$\mu = [(e/M^2) \cos^2\alpha + (f/Mr) \sin 2\alpha + (g/r^2) \sin^2\alpha)]^{1/2}$ (any direction defined by α)

Areal distortion

$P = mn \cos\varepsilon = mn \sin i = ab$ (P is area).

Maximum angular distortion (Vitkovskiy)

$\sin(\omega/2) = (a-b)/(a+b)$, $\cos(\omega/2 = (2\sqrt{(ab)}/(a+b)$, $\tan(\omega/2) = (a-b)/(2\sqrt{(ab)})$

Extreams of Local Linear Scale

Ellipsoidal extremes of local linear scale occur at the direction

$\tan 2\alpha_0 = 2mn \cos 2i / (m^2-n^2)$ (ellipsoid)

Direction in projection

$\tan 2\beta_0 = n^2 \sin 2i / (m^2+n^2\cos 2i)$

Above two equations produce two directions a_0 and $a_0 + 90^O$, b_0 and $b_0 + 90^O$ in azimuth in both ellipsoid and the projection, along those direction the extreme value of a and b (of Tissot Indicatrix) falls. These two directions are orthogonal and are called BASE DIRECTION. In conformal projection or where meridians and parallels cut at right angle, the base direction falls on them.

A study was made at Bhadrapur 26°32'N, 88°04'N to examine the deformation due to projection using Tissot Indicatrix for Bonne pseudoconical equal area projection. The study was made on scale at meridian and parallel, axis dimension a and b of tissot indicatrix ellipse, areal and angular distortion, azimuth at which the maximum deformation occur etc.

Bhadrapur

α_0 = -0.762436092 radian

β_0 = -0.762422093 radian.

At Bhadrapur the following table gives the deformation character:

a. Parallel and meridian intersect at an angle 89°59'54.22"
b. Maximum angular error = - 5' 30.89"
c. Azimuth at which the maximum over all error occur is - 43°41'3.77" (North West)
d. Base directions 43°41'3.73" and 133°41'3.73"
e. Areal scale (a * b) = 1 (equal area)

f. Scale at parallel = 1, scale at meridian 1.00000128

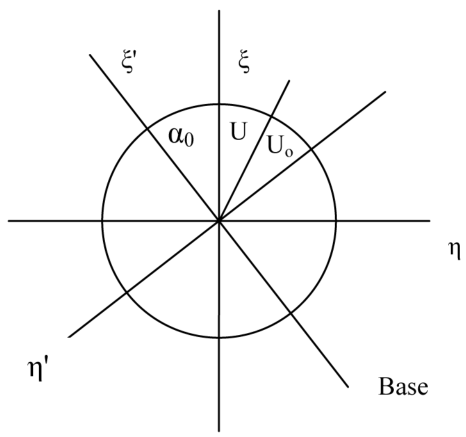
g. The ellipse axis diameter a = 1.0008024, b = 0.999198222

h. The values of angle at which angular distribution is maximum, u_o= 0.7857992092056535, v_o = 0.784997117859243.

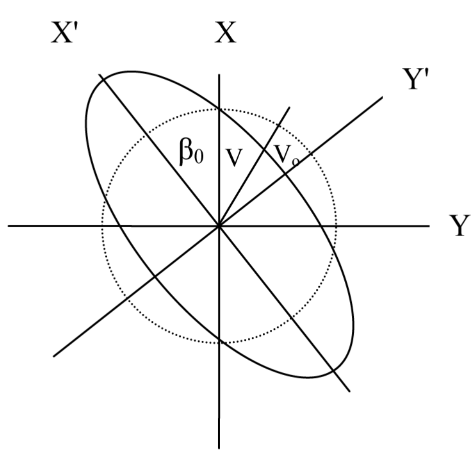
m = scale along meridian
n = scale along parallel
ai = angle of intersection between meridian and parallel
ep = deviation from right angle
omg = angular distortion
alpha = base direction in ellipsoid
beta = base direction in projection plane

BHADRAPUR (Tissot Indicatrix parameters computation)

1.000001286701646	1. 0000000000000000 m, n		
1. 000802413463986	9. 99198228871476E-001	1.00000000000000 A,B,AR	
1.604183232763055E-003	1.570768328533692	1.604182200711559E-003 OMG, AI, EP	
-1. 604182200657523E- 003	- 1.604180063880593E-003 ALPHA, BETA		
- 7. 624360922924287E- 001	- 7. 624220931708325E- 001 ALP0,B0		
7. 608319100917712E-001	7.6081791310069519E-001 U, V		
7. 857992092056535E- 001	7. 849971178592431E-001 U0, V0		



Ellipsoid



Projection Plane

alpo = extreme of local liner scale
bo = corresponding azimuth to projection plane
uo = at this direction omg will be maximum distorted (ellipsoid)
vo = at this direction omg will be maximum distorted (projection)
alpha (omg) max=alpo+uo
beta (omg) max=bo+vo

Conclusion

In Nepal UTM projection is used since 1970, due to the pattern of error distribution three grid zones have been selected. A small country like Nepal has three sets of grid zones and co-ordinate, this situation has posed problems while mosaicing all three grids at margin. Either we have to transfer the grid zone to the other grid system, however some accuracy may be lost while doing so, on the other hand transformation of grid is not a suitable means to solve the problem. Present day computer cartography, digital mapping, GIS etc. are the popular technologies which may be utilized in solving this type of problem.

Taking consideration of the geographical position of the country, its shape after projection, distortion of area, angle, azimuth etc, I came to a conclusion that projection of conic and pseudo-conic family is best suited for the country. I found conic conformal projection with a minimum scale factor of units along parallel 28^{o}N (Central parallel of Nepal) gives least areal distortion.

For equal area projection Alber equal area conic projection and Bonne Pseudo-conic equal area projection (Normal) were studied. Bonne projection gives low distortion than Alber projection.

Survey Department should setup a working group to come up with the recommendation after thorough study the problem and suggest a suitable projection for the country (Conformal, equal area, equidistant). Also technical map and general maps should be dealt separately because technical map requires high degree of accuracy and general map should give some what true picture of the country.

References

Datum and map Projection for Remote sensing GIS and Surveying, J.C.ILIFFE, Reprinted 2002

Geodosy, G.Bomford, 3rd Edition.

An introduction to differential Geometry T.J.Willmore, 7th Edition, 2002.

Map projection - A Working Manual, J.P.Snyder, US Geological Survey, Professional Paper 1395.

Map Projection Transformation Principles and Applications, Q.Yang, J.P.Snyder and R. Tobler

Map Projection Properties: Considerations for Small Scale GIS Application, E.M. Delmelle, Unpublished M.A. Thesis, U. of New York.

Map Projection used by US Geological Survey, J.P.Snyder