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Original Research Article

Modified Half-Cauchy Chen (MHCC) Distribution with Applications to Lifetime Dataset

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Abstract
In this article, we have recommended an innovative versatile distribution called Modified Half-Cauchy Chen distribution by modifying half-Cauchy Chen distribution. The recommended distribution's various properties are derived and analyzed. The recommended distribution's parameters are ascertained by applying the maximum likelihood estimation (MLE) approach. Furthermore, the performance of the Modified Half-Cauchy Chen distribution is compared against other distributions using various statistical measures. These measures consist of the Corrected Akaike Information Criterion (CAIC), the Kolmogorov-Smirnov (K-S) test, the Bayesian Information Criterion (BIC), and the Akaike Information Criterion (AIC). The results consistently demonstrate the superior fit of the Modified Half-Cauchy Chen distribution to the data. The goodness-of-fit analysis is carried out on a real data set in order to evaluate the innovative distribution's applicability. The Modified half-Cauchy Chen distribution is shown to perform better than a few other known distributions. R programming software is used to help with every computation.

Introduction
Classical probability models have been used for many years to assess real-world information in a variety of domains, including as risk analysis, biology, hydrology, geology, climatology, engineering, finance, and life testing. However, these models often fall short in providing an accurate fit. As a result, it has become essential to enhance existing distributions to better address the challenges in these areas. More flexible distributions that provide a better match than conventional classical models can be produced by altering the baseline model or adding new parameters. This approach involves enhancing the baseline models by incorporating additional parameters, which allows for greater flexibility in capturing the complexities of real-world data. To achieve this, researchers may introduce shape parameters, scale parameters, or other modifications that adjust the model's behavior to align more closely with observed data. The expansion of classical probability models through the incorporation of additional parameters represents a significant advancement in the field of statistical modeling.

When modeling monotonic hazard rates, traditional probability models like the exponential, gamma, lognormal, and Weibull distributions do not display bathtub-shaped hazard rate functions. These distributions only show hazard rates that are either monotonically increasing, decreasing, or constant. To examine real datasets with failure rates shaped like bathtubs, several parametric probability distributions have been constructed. The unique two-parameter lifespan distribution that Chen (2000) presented has an increasing failure rate (IFR) function or a bathtub-shaped failure rate. Equation (1) gives the cumulative distribution function (PDF) of this distribution.

$$G(x;\beta,\lambda)=1-\exp\left\{\lambda(1-e^{x^{\beta}})\right\};\ x>0,\ \beta,\lambda>0 \tag{1}$$

Additionally, equation (1)'s probability density function (PDF) is

$$g(x;\beta,\lambda)=\beta\lambda x^{\beta-1}e^{x^{\beta}}\exp\left\{\lambda(1-e^{x^{\beta}})\right\} \tag{2}$$

The Chen distribution fails to accurately capture or model certain survival data sets, particularly those that are skewed. Due to these shortcomings, it is necessary to modify the distribution to enhance its flexibility. The modification of the Chen distribution was inspired by the need for a more adaptable model capable of representing the diverse shapes of density and hazard functions. Dey et al. (2017) created Exponentiated Chen (EC) distribution with introducing one more shape parameter in Chen distribution which exhibits unimodal, bathtub, and increasing hazard modeling. Tarvirdizade and Ahmadpour (2019) suggested the Weibull–Chen (WC) distribution which exhibits increasing, decreasing or bathtub-shaped hazard rate function. Another extension on the Chen distribution, the Lindley-Chen distribution was developed by (Joshi & Kumar, 2020). The Chen Pareto Distribution was introduced by (Awodutire,2020). The Exponentiated Odd Chen-G family of distributions, which has a range of hazard rate functions, was created by (Eliwa et al., 2021). Depending on the parameter values, the hazard rate functions of this distribution might be unimodal-bathtub, increasing, decreasing, unimodal, bathtub, J-shaped, or inverse J-shaped. Zamani et al. (2022) developed Extended Exponentiated Chen (EE-C) distribution. Acquah et al. (2023) introduced New Extended Chen distribution which exhibits varied complex and hazard shapes. The Odd-Chen Exponential which, exhibits different shapes, including the well-known bathtub shaped hazard rate function, is a novel statistical distribution with three parameters that Otoo et al. (2023) suggested.

By reflecting the curve around the origin to concentrate exclusively on positive values, this study has investigated the half-Cauchy distribution, that created from the Cauchy distribution. This distribution, known for its pronounced tail, is useful in predicting more frequent long-distance dispersal events. Shaw (1995) employed the half-Cauchy distribution as a model for spreading distances due to its suitability for capturing such occurrences. Furthermore, ringing data for two distinct tit species in Britain and Ireland were analyzed using the half-Cauchy model by (Paradis et al.,2002).

A non-negative random variable X with a half-Cauchy distribution can have the following expression for its cumulative distribution function:

$$G(x;\theta)=\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right),\ \ x>0,\theta>0. \tag{3}$$

and equation (3)'s corresponding probability density function (PDF) is

$$g(x;\theta)=\frac{2}{\pi}\left(\frac{\theta}{\theta^{2}+x^{2}}\right),\ \ x>0,\theta>0. \tag{4}$$

In the past few decades, a number of researchers have employed the half-Cauchy distribution as a baseline. Some modifications of the half-Cauchy distribution found in literature are the Gamma half-Cauchy model (Alzaatreh et al., 2016), Generalized odd half-Cauchy family of distribution (Cordeiro et al. (2017), Kumaraswamy Half-Cauchy distribution (Ghosh, 2014),Lindley Half- Cauchy Distribution (Chaudhary & Kumar, 2022), half-Cauchy exponential extension distribution (Telee & Kumar, 2022), half-Cauchy modified exponential distribution (Chaudhary & Kumar, 2022), Half -Cauchy Exponential Distribution (Jaykumar & Fasna, 2023) and Half -Cauchy

Inverse NHE Distribution (Chaudhary et al., 2022).

Therefore, we are interested in using the half-Cauchy family of distributions to generate novel distributions. The generating family of distribution was developed by (Zografos & Balakrishnan, 2009), and its CDF may be attained as

$$F(x) = \int_0^{-\ln[1-G(x)]} r(t)\, dt\,, \tag{5}$$

The CDF of any baseline distribution in this case is represented by *G(x)*, while any distribution's PDF is represented by *r(t)*. Using formula (4)'s half-Cauchy distribution as given and its PDF, *r(t)*, as the basis, CDF of the family of half-Cauchy distributions can be expressed as

$$\begin{aligned} F(x) &= \int_0^{-\ln[1-G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\, dt \\ &= \frac{2}{\pi}\arctan\left(-\frac{1}{\theta}\ln\left[1-G(x)\right]\right);\ x>0, \theta>0 \end{aligned} \tag{6}$$

The associated PDF with (6) may be expressed as

$$f(x) = \frac{2}{\pi\theta}\frac{g(x)}{1-G(x)}\left[1+\left\{-\frac{1}{\theta}\log\left[1-G(x)\right]\right\}^2\right]^{-1} \tag{7}$$

Chaudhary et al. (2023) created innovative distribution termed as Half -Cauchy Chen (HCC) distribution by blending half-Cauchy distribution with Chen distribution as baseline model. The CDF and PDF of half -Cauchy distribution may be derived by plugging the CDF and PDF of Chen distribution from equations (1) and (2) in equations (6) and (7) respectively. Thus, CDF and PDF of Half-Cauchy Chen (HCC) may respectively be derived as

$$F(x) = \frac{2}{\pi}\arctan\left\{-\frac{\lambda}{\theta}(1-e^{x^\beta})\right\};\ x>0,\ \beta,\lambda,\theta>0 \tag{8}$$

$$f(x) = \frac{2}{\pi}\beta\lambda\theta x^{\beta-1}e^{x^\beta}\left[\theta^2+\left\{-\lambda(1-e^{x^\beta})\right\}^2\right]^{-1}. \tag{9}$$

The hazard rate function of Half- Cauchy Chen distribution is increasing, decreasing or unimodal hazard rate.

In this work, we have recommended versatile innovative distribution called Modified Half- Cauchy Chen (MHCC) to analyze the lifetime real dataset which is the extension of Chen distribution.

This is the format for the remainder of the article. We begin by defining the Modified Half-Cauchy Chen distribution and then go over some of its statistical characteristics. Next, we estimate the recommended model's parameters by means of MLE approach. Various test criteria are employed to measure the goodness of fit. Next, we use a real-life dataset to show how the suggested approach may be used. Lastly, we provide a few closing thoughts.

Modified Half Cauchy Chen (MHCC) Distribution

The Modified Half-Cauchy Chen (MHCC) distribution, which has three parameters (α, β, and θ), is a novel flexible distribution that we have recommended in this section. It is derived from the Half-Cauchy Chen distribution (Chaudhary et al., 2023) by substituting λ = 1 in equations (8) and (9), and by adding a scale parameter α. Their CDF and PDF, for a positive random variable X with an MHCC distribution, may respectively be derived as

$$F(x) = \frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}(1-e^{\alpha x^\beta})\right\};\ x>0,\ \beta,\alpha,\theta>0 \tag{10}$$

$$f(x) = \frac{2}{\pi}\alpha\beta\theta x^{\beta-1}e^{\alpha x^\beta}\left[\theta^2+\left\{-(1-e^{\alpha x^\beta})\right\}^2\right]^{-1}\ x>0,\ \beta,\alpha,\theta>0. \tag{11}$$

Reliability function

The MHCC (α, β, θ) distribution's reliability function is

$$R(x) = 1 - F(x)$$
$$= 1 - \frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}(1-e^{\alpha x^{\beta}})\right\};\ x>0,\ \beta,\alpha,\theta>0\,. \tag{12}$$

Hazard rate function (HRF)

The MHCC distribution's HRF is given by (13).

$$h(x) = \frac{f(x)}{R(x)} = 2\alpha\beta\theta x^{\beta-1}e^{\alpha x^{\beta}}\left[\pi - 2\arctan\left\{-\frac{1}{\theta}(1-e^{\alpha x^{\beta}})\right\}\right]^{-1} S, Where\ \ S = \left[\theta^2 + \left\{-(1-e^{\alpha x^{\beta}})\right\}^2\right]^{-1}. \tag{13}$$

The Reversed hazard function (RHR)

The equation (14) is the reversed hazard rate function.

$$RHR = 2\beta\theta x^{\beta-1}e^{\alpha x^{\beta}}\left[\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}(1-e^{\alpha x^{\beta}})\right\}\right]^{-1} A,\quad Where\ \mathrm{A} = \left[\theta^2 + \left\{-(1-e^{\alpha x^{\beta}})\right\}^2\right]^{-1} \tag{14}$$

Cumulative hazard function (CHF)

The equation (15) is the cumulative hazard function of the suggested model.

$$H(x) = -\log\left[1 - \frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}(1-e^{\alpha x^{\beta}})\right\}\right] \tag{15}$$

Quantile function (QF)

The equation (16) is the quantile function of suggested model.

$$Q(u) = \left[\ln\left[1+\theta\tan\left\{\frac{\pi u}{2}\right\}\right]\right]^{1/\beta};0<u<1 \tag{16}$$

where u denotes U (0,1)'s uniform random variable.

The random deviate generation of suggested model produced corresponding to equation (16) is

$$x = \left[\ln\left[1+\theta\tan\left\{\frac{\pi v}{2}\right\}\right]\right]^{1/\beta};0<v<1.$$

The MHCC distribution's median using relation (16) is

$$median = \left[\ln\left\{1+\theta\right\}\right]^{1/\beta}.$$

Skewness and Kurtosis of MHCC distribution

Using quantiles, the Bowley's coefficient of skewness may be found as

$$S = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

Using octiles, the Moors (1988)'s coefficient of kurtosis is stated as

$$K_u(M) = \frac{Q(0.875) - Q(0.125) + Q(0.375) - Q(0.625)}{Q(3/4) - Q(1/4)}.$$

The probability density function (PDF) and hazard rate function (HRF) of the MHCC model are displayed in Figure 1 below, illustrating the various shapes for various parameter values. The PDF can exhibit decreasing, positively skewed, or symmetrical shapes, while the HRF can be increasing, decreasing, or take on a reversed J-shape.

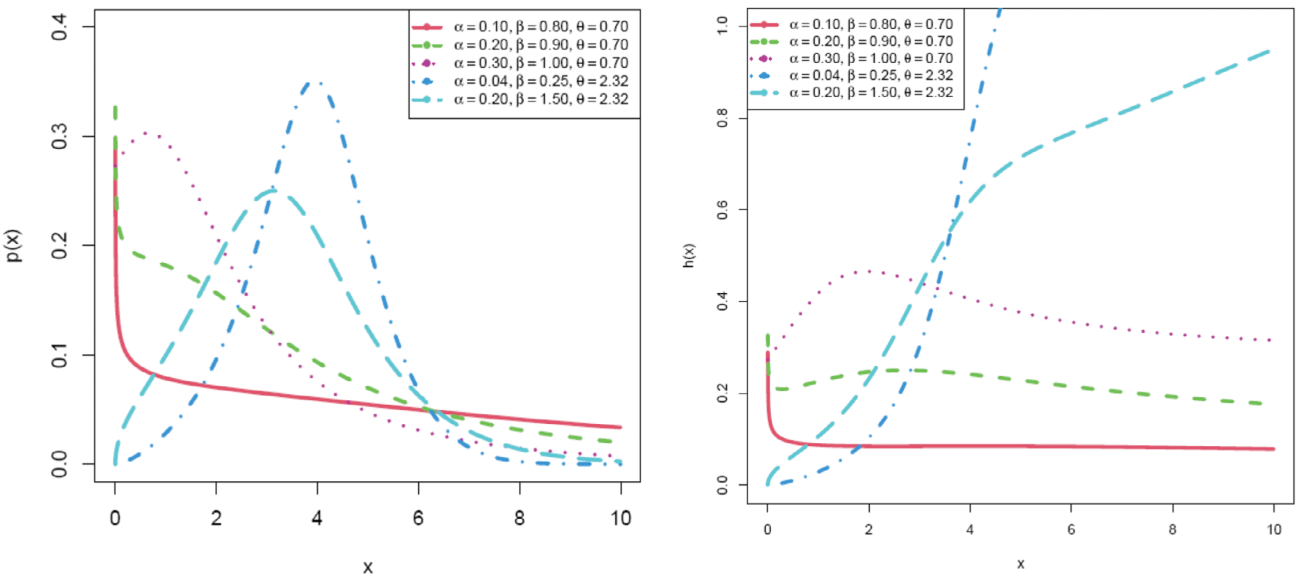


Figure 1: *PDF (left) and Hazard rate function (right) graphs for a few parameter values*

Parameter estimation

In this section of the study, we have estimated the parameters of the models proposed. there are numerous methods for parameters estimation available in theory such as; Cramer's von Mises estimation methods, Maximum likelihood estimation (MLE), Least square estimation etc. In this study, we have used MLE only.

Maximum Likelihood Estimation

The ML estimators (MLEs) for the MHCC model, which are determined by the MLE technique, are shown below. Given a random sample $\underline{x} = (x_1, \ldots, x_n)$ taken from the MHCC (α, β, θ) using the log likelihood function,

$$l(\beta,\alpha,\theta \mid \underline{x}) = n\ln(2/\pi) + n\ln(\beta\lambda\theta) + (\beta-1)\sum_{i=1}^{n}\ln x_i + \alpha\sum_{i=1}^{n}x_i^{\beta} - \sum_{i=1}^{n}\ln\left\{\theta^2 + \left\{-(1-e^{\alpha x_i^{\beta}})\right\}^2\right\}$$

Here, we can differentiate loglikelihood function with respect to unknown parameters, α, β, and θ and solved to get the estimated parameters. It is quite rigorous to estimate the parameters manually, so we have used R programming to estimate the parameters.

Real Data Analysis

To confirm the applicability of the model, we employed a real dataset comprising the remission times (in months) for a random sample of 128 bladder cancer patients (Lee & Wang, 2003). This dataset has been the subject of recent analyses in several studies, including those by [(Ramos et al. ,2015), (Hamdeni & Gasmi,2020), and (Ijaz et al.,2021)]. The values in this dataset are as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.7, 5.17, .28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83,4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07.

Table 1: *Summary Statistics*

Min.	Q1	Q2	Mean	Q3	Max.	SD	Skewness	Kurtosis
0.080	2.615	5.375	8.569	10.875	79.050	10.560	3.377	19.08611

The provided dataset exhibits positive skewness and non-normality. We also plotted the boxplot and TTT plot of the provided data set in order to investigate the model's nature, as displayed in figure 2.

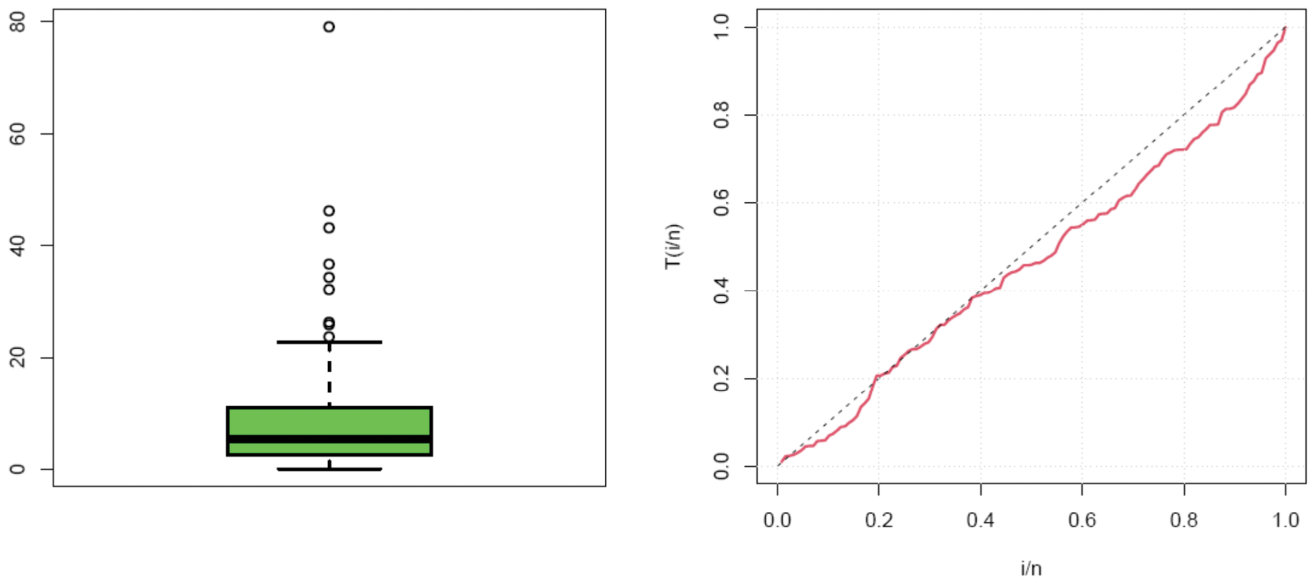


Figure 2: *Boxplot (left) and TTT plot (right) of the data considered.*

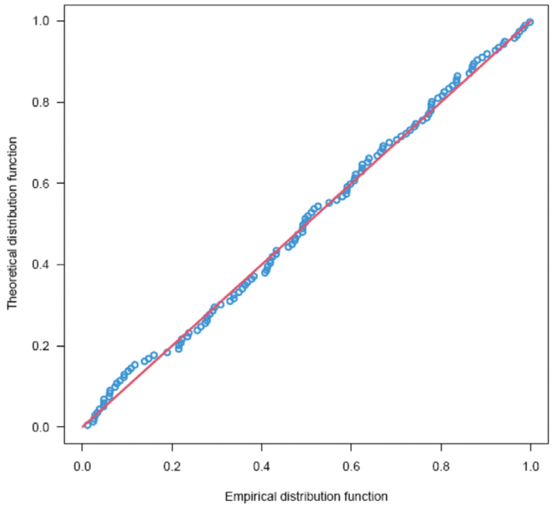
Box plot indicates that data is positively skewed with some outliers. Similarly, the TTT plot shows that the data being nearly convex and mostly below the 45-degree line, this indicates the data is from a distribution with a decreasing hazard function.

The log likelihood function specified in equation (17) is maximized to calculate the MLEs of MHCC through the usage of R software and the optim () function (R Core Team, 2023). Values determined are tabulated in Table 2.

Table 2: *Estimated parameters using MLE and SE*

Parameters	MLE	SE
alpha	0.1225716	0.1263894
beta	0.8839793	0.1846576
theta	0.7308502	0.7107335

A graph of the Q-Q and P-P plots may be seen in Figure 3. The analyzed data is found to fit the MHCC model better.



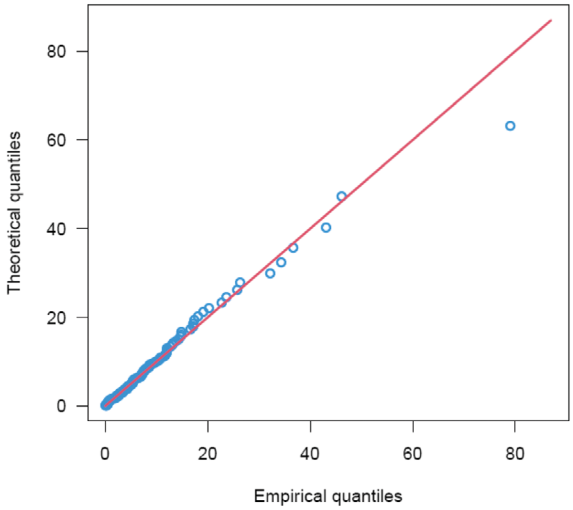


Figure 3: *The P-P plot (left) & Q-Q plot (right) of the MHCC Model.*

Table 3 lists the MHCC model's estimated parameter values with log-likelihood, AIC, BIC, CAIC, and HQIC criteria. The MHCC model's estimated parameter values with HQIC, BIC, CAIC, log-likelihood, and AIC criteria are listed in Table 3.

Table 3: *Estimated parameters with LL and values of information criteria*

LL	AIC	BIC	CAIC	HQIC	KS	W	A^2
-400.6383	807.2767	815.8328	807.4702	810.7531	0.0402(0.9859)	0.0303 (0.9756)	0.2606(0.9644)

This section of the study presents the applicability of MHCC using a real dataset that was previously used. The model's effectiveness is compared against five different distributions. The competing models are: Modified Inverse Lomax Distribution (Telee et al.,2023), Generalized Exponential Extension (GEE) distribution (Lemonte, 2013), Generalized Exponential (GE) distribution (Gupta & Kundu, 2007) and Lindley-Exponential (LE) distribution (Bhati et al. ,2015). Below is a table of various information criteria values used to assess the applicability of the MHCC.

Table 4: *CAIC, BIC, AIC, HQIC, and log likelihood Values*

Dist.	AIC.	CAIC	BIC	HQIC	LL
MHCC	807.28	807.47	815.83	810.75	-400.64
MILX	808.17	808.36	816.72	811.64	-401.08
GEE	827.20	827.39	835.76	830.68	-410.60
LE	828.09	828.19	833.80	830.42	-412.05
GE	830.16	830.25	835.86	832.47	-413.08

The MHCC model's goodness-of-fit is contrasted with that of various rival models in this section. Table 5 also shows the results of the Anderson-Darling test statistic, the Kolmogorov-Simirov test, and the Cramer-Von Mises test. The MHCC model exhibits the lowest test statistic values, indicating higher p-values and thus a better and more consistent fit compared to the other models.

Table 5: *p-values and Test statistics*

Dist.	K-S (p - Values)	W (p - Values)	A2 (p - Values)
MHCC	0.0402(0.9859)	0.0303 (0.9756)	0.2606(0.9644)
MILX	0.0433(0.9702)	0.0535(0.8556)	0.3678(0.8797)

GEE	0.0441(0.9640)	0.0393(0.9370)	0.2631(0.9630)
LE	0.0622(0.7059)	0.0899(0.6377)	0.5250(0.7211)
GE	0.0725(0.5115)	0.1279(0.4652)	0.7137(0.5472)

Conclusion

In this article, we have recommended an innovative versatile distribution called Modified Half-Cauchy Chen distribution by modifying half-Cauchy Chen distribution. Some of the novel distribution's statistical properties are explored and analyzed. The proposed model's parameters are estimated for a real data set using the widely utilized estimation technique, Maximum Likelihood Estimation (MLE). The probability density functions (PDFs) of the model's curves demonstrate its adaptability for modeling real-world data, exhibiting diverse shapes such as decreasing, positively skewed, or symmetrical. Based on the model parameters values, the hazard function can display different patterns, including increasing, decreasing, or a reversed J-shaped hazard rate. The recommended model is evaluated on an actual dataset to determine its applicability and adaptability. The findings indicate that the model is substantially more flexible than some alternative fitted distributions.

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