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GENERALIZED SASAKIAN-SPACE-FORMS WITH D-CONFORMAL CURVATURE TENSOR

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ABSTRACT

In this paper we study generalized Sasakian-space-forms with D-conformal curvature tensor. In generalized Sasakian-space-forms, we investigate some results on D-conformally flat, ξ-D-conformally flat, φ-D-conformally flat and the curvature condition $B(\xi, X).S = 0.$

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1. INTRODUCTION

A Sasakian manifold $M(\varphi,\xi,\eta,g)$ is said to be a Sasakian-space-form if all the φ-sectional curvatures $K(X\wedge\varphi X)$ are equal to a constant $c,$ where $K(X\wedge\varphi X)$ denotes the sectional curvature of the section spanned by the unit vector field $X,$ orthogonal to ξ and $\varphi X.$ In such a case, the Riemannian curvature tensor of M is given by

$$
\begin{aligned}
R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\
& + \frac{c-1}{4}\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\
& + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
& \qquad - g(Y,Z)\eta(X)\xi\}.
\end{aligned}
\tag{1.1}
$$

These spaces can be modeled, depending on $c > -3, c = -3$ or $c < -3.$

As a natural generalization of these manifolds Alegre, Blair and Carriazo introduced and studied the notion of generalized Sasakian-space-forms in 2004 [1]. They replaced constant quantities $(c+3)/4$ and $(c-1)/4$ of relation (1.1) by differentiable functions f_1, f_2 and $f_3.$

An almost contact metric manifold $M(\varphi,\xi,\eta,g)$ is said to be a generalized Sasakian-space-form if the curvature tensor R is given by [1]

(1.2)
$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & \quad - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. In such a case manifold is denoted by $M(f_1, f_2, f_3)$. In [2] authors studied contact metric and trans-Sasakian generalized Sasakian-space-forms. In [5] and [6] authors studied on the locally φ-symmetric and η-recurrent Ricci tensor and on the projective curvature tensor respectively. Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms were studied by Kim [7]. Shukla and Shah studied on generalized Sasakian-space-forms with concircular curvature tensor [8].

A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies [9]

$$R(X,Y).R = 0, \qquad X, Y \in TM,$$

where $R(X,Y)$ acts on R as a derivation.

Generalized Sasakian-space-forms have also been studied by [10] and others.

2. PRELIMINARIES

In an almost contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, where φ is a (1, 1) tensor field, ξ is a contravariant vector field, η is a 1-form and g is a compatible Riemannian metric, we have [3]

(2.1) $\varphi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta(\varphi X) = 0,$

(2.2) $g(X,\xi) = \eta(X),$

(2.3) $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$

(2.4) $g(\varphi X, Y) = -g(X, \varphi Y),$

(2.5) $(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).$

In a $(2n+1)-$dimensional generalized Sasakian-space-form the following relations hold:

(2.6) $R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\},$

(2.7) $R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\},$

(2.8) $R(\xi,X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\},$

(2.9) $$S(X,Y)=(2nf_1+3f_2-f_3)g(X,Y)-(3f_2+(2n-1)f_3)\eta(X)\eta(Y),$$

(2.10) $$r=2n(2n+1)f_1+6nf_2-4nf_3,$$

(2.11) $$S(X,\xi)=2n(f_1-f_3)\eta(X),$$

(2.12) $$\eta(R(X,Y)Z)=(f_1-f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\},$$

(2.13) $$\eta(R(X,Y)\xi)=0,$$

(2.14) $$\eta(R(\xi,X)Y)=(f_1-f_3)\{g(X,Y)-\eta(X)\eta(Y)\},$$

(2.15) $$S(\varphi X,\varphi Y)=S(X,Y)-2n(f_1-f_3)\eta(X)\eta(Y).$$

The D-conformal curvature tensor on a Riemannian manifold (M^{2n+1},g) is defined as [4]

(2.16)
$$\begin{aligned}B(X,Y)Z={}&R(X,Y)Z+\frac{1}{2(n-1)}[S(X,Z)Y-S(Y,Z)X\\&+g(X,Z)QY-g(Y,Z)QX-S(X,Z)\eta(Y)\xi\\&+S(Y,Z)\eta(X)\xi-\eta(X)\eta(Z)QY+\eta(Y)\eta(Z)QX]\\&-\frac{k-2}{2(n-1)}[g(X,Z)Y-g(Y,Z)X]\\&+\frac{k}{2(n-1)}[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\\&+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X]\end{aligned}$$

where $k=\frac{r+4n}{2n-1}$, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

3. RESULTS AND DISCUSSION

Definition: A $(2n+1)-$dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is said to be D-conformally flat if

(3.1) $$B(X,Y)Z=0.$$

Theorem 3.1. *If a $(2n+1)-$dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is D-conformally flat, then $f_3=f_1+1$.*

Proof. Let us consider a $(2n+1)-$dimensional generalized Sasakian-space-form which satisfies the condition $B(X,Y)Z=0,$ then from (2.16) we have

(3.2)
$$\begin{aligned}0 = R(X,Y)Z + \frac{1}{2(n-1)}[&S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi \\ &- \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ &- \frac{k-2}{2(n-1)}[g(X,Z)Y - g(Y,Z)X] \\ &+ \frac{k}{2(n-1)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\qquad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].\end{aligned}$$

Taking inner product on both sides of (3.2) by $W,$ we get

(3.3)
$$\begin{aligned}0 = R(X,Y,Z,W) + \frac{1}{2(n-1)}[&S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &+ S(Y,W)g(X,Z) - S(X,W)g(Y,Z) - S(X,Z)\eta(Y)\eta(W) \\ &+ S(Y,Z)\eta(X)\eta(W) - S(Y,W)\eta(X)\eta(Z) + S(X,W)\eta(Y)\eta(Z)] \\ &- \frac{k-2}{2(n-1)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ &+ \frac{k}{2(n-1)}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &\qquad + g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)]\end{aligned}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W).$

Setting $W = \xi$ in (3.3) and using (2.1) and (2.2), we obtain

(3.4)
$$\begin{aligned}0 = \eta(R(X,Y)Z) + \frac{1}{2(n-1)}[&S(Y,\xi)g(X,Z) \\ &- S(X,\xi)g(Y,Z) - S(Y,\xi)\eta(X)\eta(Z) + S(X,\xi)\eta(Y)\eta(Z) \\ &+ 2\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}].\end{aligned}$$

Using (2.11) and (2.12) in (3.4), we get

(3.5)
$$\left(\frac{f_3 - f_1 - 1}{n-1}\right)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} = 0.$$

Since $g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \neq 0,$ we must have $f_3 - f_1 - 1 = 0,$ this implies that

(3.6)
$$f_3 = f_1 + 1.$$

This completes the proof of the theorem.

Definition. Generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension $(2n+1)$ is said to be ξ-D-conformally flat if

(3.7) $$B(X,Y)\xi = 0.$$

Theorem 3.2. *If a* $(2n+1)-$*dimensional generalized Sasakian-space-form* $M(f_1, f_2, f_3)$ *satisfies the condition* $B(X,Y)\xi = 0,$ *then* $f_3 = f_1 + 1.$

Proof. Suppose the condition $B(X,Y)\xi = 0$ holds in a $(2n+1)-$dimensional generalized Sasakian-space-form. Then using (2.1) and (2.2) in (2.16), we have

(3.8) $$0 = R(X,Y)\xi + \frac{1}{2(n-1)}[S(X,\xi)Y - S(Y,\xi)X - S(X,\xi)\eta(Y)\xi + S(Y,\xi)\eta(X)\xi] + 2[\eta(X)Y - \eta(Y)X].$$

In view of (2.6) and (2.11), (3.8) reduces to

(3.9) $$\left(\frac{f_3 - f_1 - 1}{n-1}\right)\{\eta(Y)X - \eta(X)Y\} = 0.$$

Since $\eta(Y)X - \eta(X)Y \neq 0,$ we must have $f_3 - f_1 - 1 = 0,$ this implies that

(3.10) $$f_3 = f_1 + 1.$$

Hence the theorem is proved.

From theorem 3.1 and theorem 3.2 we obtain the following:

Corollary 3.1. In a (2n+1)-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ the curvature conditions $B(X,Y)Z = 0$ and $B(X,Y)\xi = 0$ are equivalent.

Definition. Let $M(f_1, f_2, f_3)$ be a $(2n+1)-$dimensional generalized Sasakian-space-form. Then $M(f_1, f_2, f_3)$ is said to be φ-D-conformally flat if

(3.11) $$g(B(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0.$$

Theorem 3.3. *If a* $(2n+1)-$*dimensional generalized Sasakian-space-form is* φ*-D-conformally flat, then it is an* η*-Einstein manifold under the condition* $Tr.\varphi = 0.$

Proof. Let $M(f_1, f_2, f_3)$ be a $(2n+1)-$dimensional generalized Sasakian-space-form. Suppose $M(f_1, f_2, f_3)$ satisfies the condition $g(B(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$ then from (2.1) and (2.16), we have

(3.12)

$$
\begin{aligned}
g\left(B(\varphi X,\varphi Y)\varphi Z,\varphi W\right) &= g\left(R(\varphi X,\varphi Y)\varphi Z,\varphi W\right)\\
&+\frac{1}{2(n-1)}[S(\varphi X,\varphi Z)\,g(\varphi Y,\varphi W)-S(\varphi Y,\varphi Z)\,g(\varphi X,\varphi W)\\
&+S(\varphi Y,\varphi W)\,g(\varphi X,\varphi Z)-S(\varphi X,\varphi W)\,g(\varphi Y,\varphi Z)]\\
&-\frac{k-2}{2(n-1)}[g(\varphi X,\varphi Z)\,g(\varphi Y,\varphi W)-g(\varphi Y,\varphi Z)\,g(\varphi X,\varphi W)].
\end{aligned}
$$

In view of (3.11) and (3.12), we get

$$
\begin{aligned}
0 &= g\left(R(\varphi X,\varphi Y)\varphi Z,\varphi W\right)+\frac{1}{2(n-1)}[S(\varphi X,\varphi Z)\,g(\varphi Y,\varphi W)\\
&-S(\varphi Y,\varphi Z)\,g(\varphi X,\varphi W)+S(\varphi Y,\varphi W)\,g(\varphi X,\varphi Z)\\
&-S(\varphi X,\varphi W)\,g(\varphi Y,\varphi Z)]\\
&-\frac{k-2}{2(n-1)}[g(\varphi X,\varphi Z)\,g(\varphi Y,\varphi W)\\
&-g(\varphi Y,\varphi Z)\,g(\varphi X,\varphi W)]. \qquad (3.13)
\end{aligned}
$$

By virtue of (1.2), (2.3), (2.4) and (2.15), (3.13) yields

$$
\begin{aligned}
0 &= f_1\{g(Y,Z)\,g(X,W)-g(Y,Z)\eta(X)\eta(W)-g(X,W)\eta(Y)\eta(Z)\\
&-g(X,Z)\,g(Y,W)+g(Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)\}\\
&+f_2\{g(X,\varphi Z)\,g(\varphi Y,W)-g(\varphi Y,Z)\,g(X,\varphi W)\\
&+2g(X,\varphi Y)\,g(\varphi Z,W)\}+\frac{1}{2(n-1)}[S(X,Z)\,g(Y,W)\\
&-S(X,Z)\eta(Y)\eta(W)-2n(f_1-f_3)\,g(Y,W)\eta(X)\eta(Z)\\
&-S(Y,Z)\,g(X,W)+S(Y,Z)\eta(X)\eta(W)+2n(f_1-f_3)\,g(X,W)\eta(Y)\eta(Z)\\
&+S(Y,W)\,g(X,Z)-S(Y,W)\eta(X)\eta(Z)-2n(f_1-f_3)\,g(X,Z)\eta(Y)\eta(W)\\
&-S(X,W)\,g(Y,Z)+S(X,W)\eta(Y)\eta(Z)+2n(f_1-f_3)\,g(Y,Z)\eta(X)\eta(W)]\\
&-\frac{k-2}{2(n-1)}[g(X,Z)\,g(Y,W)-g(X,Z)\eta(Y)\eta(W)-g(Y,W)\eta(X)\eta(Z)\\
&-g(Y,Z)\,g(X,W)+g(Y,Z)\eta(X)\eta(W)+g(X,W)\eta(Y)\eta(Z)]. \qquad (3.14)
\end{aligned}
$$

Let $\{e_i :\ i=1,2,.........,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = W = e_i$ in (3.14) and taking summation over $i,\ 1 \le i \le 2n+1,$ we get

(3.15)
$$\begin{aligned} 0 = &(2n-1) f_1 \{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &+ f_2 \{3g(\varphi Y, \varphi Z) - g(\varphi Y, Z) Tr.\varphi\} \\ &+ \frac{1}{2(n-1)}[-2(n-1)S(Y,Z) \ - S(Z,\xi)\eta(Y) \\ &+ \{2n(2n-1)(f_1 - f_3) + r\}\eta(Y)\eta(Z) \\ &-S(Y,\xi)\eta(Z) + \{2n(f_1 - f_3) - r\} g(Y,Z)] \\ &+ \frac{r+2}{2(n-1)}[g(Y,Z) - \eta(Y)\eta(Z)]. \end{aligned}$$

By the use of (2.3) and (2.11), (3.15) reduces to

(3.16)
$$\begin{aligned} S(Y,Z) = &\left[\frac{(2n^2 - 2n + 1) f_1 + 3(n-1) f_2 - nf_3 + 1}{n-1}\right] g(Y,Z) \\ &+ \left[\frac{n(3-2n) f_3 - f_1 - 3(n-1) f_2 - 1}{n-1}\right]\eta(Y)\eta(Z), \end{aligned}$$

under the condition $Tr.\varphi = 0.$

From (3.16) we get

(3.17) $$S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z),$$

where $\alpha = \dfrac{(2n^2 - 2n + 1) f_1 + 3(n-1) f_2 - nf_3 + 1}{n-1}$ and

$$\beta = \frac{n(3-2n) f_3 - f_1 - 3(n-1) f_2 - 1}{n-1}.$$

The relation (3.17) implies that the manifold is an η-Einstein manifold. This completes the proof of the theorem.

Theorem 3.4. *A* $(2n+1)$*-dimensional generalized Sasakian-space-form* $M(f_1, f_2, f_3)$ *satisfying the condition* $B(\xi, X).S = 0$ *is an Einstein manifold and has the scalar curvature* $r = 2n(2n+1)(f_1 - f_3).$

Proof. Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Suppose that $M(f_1, f_2, f_3)$ satisfies the condition $(B(\xi, X).S)(U,V) = 0,$ where S is the Ricci tensor. Then we have

(3.18) $$S(B(\xi, X)U, V) + S(U, B(\xi, X)V) = 0.$$

In view of (2.1), (2.2), (2.7) and (2.11), (2.16) yields

$$\begin{aligned} B(\xi,Y)Z =& \left(f_1 - f_3 - \frac{1}{n-1}\right)\{g(Y,Z)\xi - \eta(Z)Y\} \\ &+ \frac{1}{2(n-1)}[2n(f_1 - f_3)\{Y - \eta(Y)\xi\}\eta(Z) \\ &- \{g(Y,Z) - \eta(Y)\eta(Z)\}Q\xi]. \end{aligned} \tag{3.19}$$

Using (3.19) in (3.18) we get

$$\begin{aligned} 0 =& \left(f_1 - f_3 - \frac{1}{n-1}\right)\{2n(f_1 - f_3)g(X,U)\eta(V) - S(X,V)\eta(U)\} \\ &+ \frac{n(f_1 - f_3)}{n-1}[\{S(X,V) - 2n(f_1 - f_3)\eta(X)\eta(V)\}\eta(U) \\ &- 2n(f_1 - f_3)\{g(X,U) - \eta(X)\eta(U)\}\eta(V)] \\ &+ \left(f_1 - f_3 - \frac{1}{n-1}\right)\{2n(f_1 - f_3)g(X,V)\eta(U) - S(X,U)\eta(V)\} \\ &+ \frac{n(f_1 - f_3)}{n-1}[\{S(X,U) - 2n(f_1 - f_3)\eta(X)\eta(U)\}\eta(V) \\ &- 2n(f_1 - f_3)\{g(X,V) - \eta(X)\eta(V)\}\eta(U)]. \end{aligned} \tag{3.20}$$

Putting $V = \xi$ in (3.20) and using (2.1), (2.2) and (2.11), we obtain

$$S(X,U) = 2n(f_1 - f_3)g(X,U). \tag{3.21}$$

The relation (3.21) implies that the generalized Sasakian-space-form is an Einstein manifold.

Again, taking an orthonormal frame field at any point of the manifold and contracting over X and U in (3.21) we have

$$r = 2n(2n+1)(f_1 - f_3), \tag{3.22}$$

where r is the scalar curvature.

In view of (3.21) and (3.22), the theorem is proved.

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