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ON THE 3rd ORDER LINEAR DIFFERENTIAL EQUATION

[1]B. L. Moreno-Ley, [1]J. López-Bonilla*, [2]B. Man Tuladhar

[1]ESIME-Zacatenco-ICE, Instituto Politécnico Nacional, CP 07738 México DF
[2]Kathmandu University, Dhulikhel, Kavre, Nepal,

*Corresponding address: jlopezb@ipn.mx
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ABSTRACT
If for an arbitrary 3th order linear differential equation, non-homogeneous, we know two solutions of its associated homogeneous equation (HE), then we show how to determine the third solution of HE and the particular solution of the original equation.

Keywords: Wronskian, Linear differential equations, Method of variation of parameters

INTRODUCTION

If for the linear differential equation of third order:

$$p(x)y'' + q(x)y' + r(x)y = \Phi(x), \tag{1}$$

we know the solution y_1 of the corresponding homogeneous equation (HE):

$$p\, y'' + q\, y' + r\, y = 0, \tag{2}$$

then it is possible to obtain the solution y_2 of (2) and the particular solution y_p of (1) [1-5]:

$$y_2(x) = y_1(x)\int^x \frac{\tilde{w}}{y_1^2}\, d\eta\,, \quad y_p(x) = y_2(x)\int^x \frac{y_1\,\phi}{p\,\tilde{w}}\, d\eta - \quad y_1(x)\int^x \frac{y_2\,\phi}{p\,\tilde{w}}\, d\eta\;, \tag{3}$$

where $\tilde{W}$ is the Wronskian of the two independent solutions of (2), with the Abel – Liouville – Ostrogradski identity:

$$\tilde{W} \equiv\; y_1\, y_2' - y_2\, y_1' = \exp\,\left(-\int^x \frac{q}{p}\, d\xi\,\right) \tag{4}$$

The expression (3) for y_p can be constructed via method of variation of parameters of Euler (1741) – Lagrange (1777), or employing the technique of adjoint-exact linear differential operator[4, 5].

Here we consider the differential equation of third order:

$$u(x)y''' + p(x)y'' + q(x)y' + r(x)y = \phi(x)\,, \qquad (5)$$

and we accept the knowledge of the solutions y_1 & y_2 of its HE:

$$u\,y''' + p\,y'' + q\,y' + r\,y = 0\,, \qquad (6)$$

with the aim to find expressions for the particular solution of (5) and the solution y_3 of (6).

THIRD ORDER LINEAR DIFFERENTIAL EQUATION

In this case, the HE (6) has three solutions:

$$u\,y_j''' + p\,y_j'' + q\,y_j' + r\,y_j = 0\,, \qquad j = 1, 2, 3 \qquad (7)$$

whose linear independence implies a non-null Wronskian :

$$W \equiv \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}. \qquad (8)$$

The derivative of (8) gives:

$$\frac{dW}{dx} = y_1'''\,W_{23} + y_2'''\,W_{31} + y_3'''\,W_{12}, \qquad (9)$$

with the notation:

$$W_{i\,j} = -\,W_{j\,i} = y_i\,y_j' - y_j\,y_i'\,, \quad i \neq j. \qquad (10)$$

If (9) is multiplied by u(x) and we use (7), then:

$$u\,\frac{d\,W}{d\,x} = -p\,W \quad \therefore \quad W = k\exp\left(-\int^x \frac{p}{u}\,d\xi\right),$$

but, without loss of generality, we may take k=1 because we can multiply the y_j by an adequate scale factor (they are solutions of a HE), therefore:

$$W = \exp\left(-\int^{x} \frac{p}{u}\, d\,\xi\right) \,, \tag{11}$$

is the Abel – Liouville – Ostrogradski identity for (5).

The expansion of the determinant (8), via the third column, implies:

$$W_{12}\, y_3'' - W_{12}'\, y_3' + (\, y_1'\, y_2'' - \; y_2'\, y_1''\,)\, y_3 = W \,, \tag{12}$$

where, in accordance with (10):

$$W_{12} = y_1\, y_2' - y_2\, y_1' \quad , \qquad W_{12}' = \frac{d}{dx} W_{12} = y_1\, y_2'' - y_2\, y_1'' \;. \tag{13}$$

It is interesting to see that y_3 satisfies the HE (6) of 3th order, and besides it is a particular solution of the non-homogeneous equation (12) of 2th order. It is simple to verify that y_1 & y_2 are solutions of the HE of (12):

$$W_{12}\, y_c'' - W_{12}'\, y_c' + (y_1'\, y_2'' - \; y_2'\, y_1'')\, y_c = 0 \;, \qquad c = 1, 2 \;, \tag{14}$$

then the method of variation of parameters gives the particular solution for (12):

$$y_3(x) = y_2(x) \int^{x} \frac{y_1\, W}{(W_{12})^2}\, d\eta \; - \; y_1(x) \int^{x} \frac{y_2\, W}{(W_{12})^2}\, d\eta \;, \tag{15}$$

thus y_3 is determined employing y_1 & y_2 .

With (8) and (10) it is easy to prove the identities:

$$\begin{aligned} y_1\; W_{23} + y_2\; W_{31} + y_3\, W_{12} &= 0 \,, \\ y_1'\; W_{23} + \; y_2'\, W_{31} + y_3'\; W_{12} &= 0 \,, \\ y_1''\, W_{23} + \; y_2''\, W_{31} + y_3''\, W_{12} &= W \,, \end{aligned} \tag{16}$$

which permit to construct the particular solution of (5):

$$y_p(x) = y_1(x)\int^x \frac{W_{23}}{W}\,\frac{\phi}{u}\,d\eta + y_2(x)\int^x \frac{W_{31}}{W}\,\frac{\phi}{u}\,d\eta + y_3(x)\int^x \frac{W_{12}}{W}\,\frac{\phi}{u}\,d\eta\,, \qquad (17)$$

with W given by (11).

The relations (15) and (17) are the generalizations of (3) for the 3th order case, and they are not explicitly given in the literature.

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