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APPROXIMATION OF A FUNCTION f OF Lip (ξ (t), p) CLASS BY (C,1) (N, p_n) METHOD OF ITS FOURIER SERIES

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ABSTRACT
In this paper, the degree of approximation of a function belonging to Lip (ξ(t),p) class by product summability method (C,1) (N, p_n) of its Fourier series has been determined.

Keywords: Degree of approximation, Fourier series, Lip (ξ(t), p) class, (C,1)(N, p_n) summability means.

AMS Classification. 41A35, 40G05.

INTRODUCTION
Bernstein [2], Alexits [1], Sahney and Goel [8] and Chandra [3] have determined the degree of approximation of a function belonging to Lipα class by (C, 1), (C, δ), (N, p_n) and $(\tilde{N}, p_n)$ means of its Fourier series. Working in same direction, Sahney & Rao [9] and Khan [4] have studied the degree of approximation of functions belonging to Lip (α, p) by (N, p_n) and (N, p, q) means respectively. Working in same direction Qureshi [7] and Qureshi and Nema ([5],[6]) have studied the degree of approximation of a function of class Lip α, Lip (α,p) and Lip (ξ (t), p) by (N,p_n) summability means. But till now nothing seems to have been done to obtain the degree of approximation of function belonging to Lip(ξ(t),p) by product summability method (C,1)(N,p_n). (C,1) (C,δ), $\delta > 0$ is particular case of (C,1) (N,p_n) summability method. An attempt to make an advance study in this direction, in present paper, the degree of approximation of a function f belonging to Lip (ξ (t), p) by (C,1) (N,p_n) means of its Fourier series has been determined.

DEFINITIONS AND NOTATIONS
Let f be 2π periodic function, Lebesgue integrable and a function of Lip(ξ(t),p), Fourier series of f(x) is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

We define the norm $\| \ \|_p$ by

$$\| f \|_p = \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{\frac{1}{p}}, \qquad p \geq 1.$$

The degree of approximation $E_n(f)$ of function f: R$\rightarrow$R is given by, (Zygmund [11])

$$E_n(f) = \operatorname{Min} \| t_n - f \|_p$$

where t_n is trigonometrical polynomial of degree n.

A function $f(x) \in \text{Lip}\alpha$ if

$$|f(x+t) - f(x)| = O\left(|t|^{\alpha}\right) \text{ for } 0 < \alpha \leq 1.$$

$f(x) \in \text{Lip}(\alpha, p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx\right)^{\frac{1}{p}} = O\left(|t|^{\alpha}\right), \; 0 < \alpha \leq 1, \; p \geq 1.$$

Given a positive increasing function $\xi(t)$, $p \geq 1$,
$f(x) \in \text{Lip}(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx\right)^{\frac{1}{p}} = O(\xi(t))$$

If $\xi(t) = t^{\alpha}$, $\text{Lip}(\xi(t), p)$ coincides with the class $\text{Lip}(\alpha, p)$ and If $p \to \infty$ in the $\text{Lip}(\alpha, p)$ class, then $\text{Lip}(\alpha, p)$ reduces to $\text{Lip}\,\alpha$ class.

Let $\sum_{n=0}^{\infty} u_n$ be infinite series whose n^{th} partial sum $S_n = \sum_{\upsilon=0}^{n} u_{\upsilon}$.

Cesåro mean (C,1) of sequence $\{S_n\}$ is defined by

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k .$$

If $\sigma_n \to S$, as $n \to \infty$ then the sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesåro mean (C,1) to S. It is denoted by

$$\sigma_n \to S((C,1)), \text{ as } n \to \infty.$$

Let $\{p_n\}$ be sequence of positive real constant such that $P_n = \sum_{i=0}^{n} p_i$ and $P_{-1} = p_{-1} = 0$. Nörlund mean (N, p_n) of sequence $\{S_n\}$ is

$$t_n^p = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \, S_k .$$

Here , If $t_n^p \to S$, as $n \to \infty$ then the sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Nörlund mean (N, p_n) to S. It is denoted by

$$t_n^p \to S((N, p_n)), \text{ as } n \to \infty.$$

If the method of summability (C,1) is superimposed on Nörlund mean (N, p_n), another method of summability (C,1) (N, p_n) is obtained.

We write

$$t_n^{c_1, p_n} = \frac{1}{n+1} \sum_{k=0}^{n} t_k^p = \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} S_r \right).$$

Here (C,1) (N, p_n) means of sequence $\{S_n\}$ define sequence $\left\{t_n^{c_1, p_n}\right\}$.

If $t_n^{c_1, p_n} \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C,1) (N, p_n) method to S. It is denoted by

$$t_n^{c_1, p_n} = S((C,1)\,(N, p_n)), \quad \text{as } n \to \infty.$$

We shall use following notations

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) \qquad (2)$$

$$N_n^{c_1,p_n}(t) = \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right). \qquad (3)$$

MAIN THEOREM

A quite good amount of works are known for approximation of a function f belonging to $Lip(\xi(t),p)$ class by (C,1), (C,δ) δ >0 and (N, p_n) summability method. Object of this paper is to study the approximation of f belonging to $Lip(\xi(t),p)$ class by product summability method of the form (C,1) (N, p_n). In fact, in this paper, we prove the following theorem;

Theorem. If f: R $\rightarrow$ R is 2π periodic and Lebesgue integrable, belonging to $Lip(\xi(t),p)$ class, then the degree of approximation of f by (C,1)(N,p_n) summability means

$t_n^{c_1,p_n} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} s_r$ of Fourier series (1) is given by

$$\left\| t_n^{c_1,p_n} - f \right\|_p = O\left((n+1)^{\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right),$$

provided $\left\{\frac{\xi(t)}{t}\right\}$ is monotonic decreasing and $\xi(t)$ satisfy the following conditions;

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right), \qquad (4)$$

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left((n+1)^{\delta}\right) \qquad (5)$$

where δ is an arbitrary number such that q(1-δ)-1> 0, condition (4) and (5) hold uniformly in x.

LEMMAS

We need the following Lemmas for the proof of the theorem.

Lemma I: If $N_n^{c_1,p_n}(t)$ is given by (3) then,

$N_n^{c_1,p_n}(t) = O(n+1)$, for $0 < t < \frac{1}{n+1}$

Proof: For $0 < t < \frac{1}{n+1}$, $\left|\sin\left(r+\frac{1}{2}\right)t\right| \le (2r+1)\left|\sin\frac{t}{2}\right|$,

$$\left| N_n^{c_1,p_n}(t) \right| = \left| \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{t}{2}\right)t}{\sin\frac{t}{2}} \right) \right|$$

$$\le \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r}(2r+1)$$

$$\le \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \frac{(2k+1)}{P_k} \sum_{r=0}^{k} p_{k-r}$$

$$= \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} (2k+1)$$
$$= \frac{1}{2(n+1)\pi} \left[\frac{2\, n(n+1)}{2} + (n+1) \right]$$
$$= \frac{(n+1)}{2\pi}$$
$$= O\,(n+1)\,. \qquad (6)$$

Lemma II: If $N_n^{c_1,p_n}(t)$ is given by (3) then,

$$N_n^{c_1,p_n}(t) = O\left(\frac{1}{t}\right), \quad \text{for } \frac{1}{n+1} < t < \pi$$

Proof: For $\frac{1}{n+1} < t < \pi$, $\left|\sin\left(r+\frac{1}{2}\right) t\right| \le 1,\quad \sin\theta \ge \frac{2\theta}{\pi},\quad 0<\theta<\frac{\pi}{2}$,

$$\left|N_n^{c_1,p_n}(t)\right| = \left|\frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right)\right|$$
$$\le \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\left|\sin\left(r+\frac{t}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|} \right)$$
$$\le \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\pi}{t} \right)$$
$$= \frac{1}{2t(n+1)} \sum_{k=0}^{n} 1$$
$$= O\left(\frac{1}{t}\right). \qquad (7)$$

PROOF OF THE THEOREM

Following Titchmarsh [10], n^{th} partial sum $S_n(x)$ of Fourier series (1) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\phi(t)\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

The (N,p_n) transform t_n^p of $\{S_n(x)\}$ is given by

$$t_n^p(x) - f(x) = \frac{1}{2\pi P_n} \int_0^{n} \phi(t) \sum_{k=0}^{n} p_{n-k} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

The (C,1) (N,p_n) transform $t_n^{c_1,p_n}$ of $\{S_n(x)\}$ i.e. (C,1) transform of $\{t_n^{p_n}\}$ is given by

$$\frac{1}{n+1} \sum_{k=0}^{n} \left(t_n^p(x) - f(x)\right) = \int_0^{\pi} \phi(t) \frac{1}{2(n+1)\pi} \sum_{k=0}^{n} \left(\frac{1}{P_k} \sum_{r=0}^{k} p_{k-r} \frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right) dt$$

$$t_n^{c_1,p_n}(x) - f(x) = \int_0^{\pi} \phi(t)\, N_n^{c_1,p_n}(t)\, dt$$
$$= \int_0^{\frac{1}{n+1}} \phi(t)\, N_n^{c_1,p_n}(t)\, dt + \int_{\frac{1}{n+1}}^{\pi} \phi(t)\, N_n^{c_1,p_n}(t)\, dt$$
$$= I_1 + I_2\,. \qquad (8)$$

Let us consider I_1.

$$|I_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)| \left| N_n^{c_1,p_n}(t) \right| dt .$$

Applying Holder's inequality and fact that $\phi(t) \in Lip(\xi(t),p)$, we have

$$\le \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t) N_n^{c_1,p_n}(t)}{t} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O\left(\frac{1}{n+1} \right) O(n+1) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)}{t} \right)^q dt \right\}^{\frac{1}{q}}$$, by condition (4) and Lemma I

$$= O\left(\xi\left(\frac{1}{n+1} \right) \right) \left\{ \int_{\epsilon}^{\frac{1}{n+1}} t^{-q} dt \right\}^{\frac{1}{q}}$$, for some $0 < \epsilon < \frac{1}{n+1}$, by Second Mean Value Theorem for Integrals.

$$= O\left(\xi\left(\frac{1}{n+1} \right) \right) \left[\left(\frac{t^{-q+1}}{-q+1} \right)_{\epsilon}^{\frac{1}{n+1}} \right]^{\frac{1}{q}}$$

$$= O\left((n+1)^{1-\frac{1}{q}} \xi\left(\frac{1}{n+1} \right) \right)$$

$$= O\left((n+1)^{\frac{1}{p}} \xi\left(\frac{1}{n+1} \right) \right). \quad \left(\because \frac{1}{p} + \frac{1}{q} = 1 \right). \tag{9}$$

Let us consider I_2.

Applying Holder's inequality, and taking δ as an arbitrary number such that $q(1-\delta) - 1 > 0$, we have

$$|I_2| \le \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t) N_n^{c_1,p_n}(t)}{t^{-\delta}} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta} \right) \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{1-\delta}} \right)^q dt \right\}^{\frac{1}{q}}$$, by condition (5) and lemma II

$$= O\left((n+1)^{\delta} \right) \left\{ \int_{\frac{1}{\pi}}^{n+1} \left(\frac{\xi\left(\frac{1}{y} \right)}{y^{\delta-1}} \right)^q \frac{dy}{y^2} \right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta} \right) O\left(\xi\left(\frac{1}{n+1} \right) \right) \left\{ \int_{\frac{1}{\pi}}^{n+1} \frac{dy}{y^{\delta q - q + 2}} \right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\ \xi\left(\frac{1}{n+1}\right)\right)\ \left\{\left[\frac{y^{q(1-\delta)-1}}{q(1-\delta)-1}\right]_{\frac{1}{\pi}}^{n+1}\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\ \xi\left(\frac{1}{n+1}\right)\right) O\left(\frac{1}{(n+1)^{\delta-1+\frac{1}{q}}}\right)$$

$$= O\left((n+1)^{1-\frac{1}{q}}\ \xi\left(\frac{1}{n+1}\right)\right)$$

$$= O\left((n+1)^{\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right). \qquad (10)$$

Collecting (8), (9) and (10) we have,

$$\left|t_n^{c_1,p_n}(x) - f(x)\right| = O\left((n+1)^{\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right),$$

or, $$\left\|t_n^{c_1,p_n}(x) - f(x)\right\|_p = O\left[\int_0^{2\pi}\left\{(n+1)^{\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}^p dx\right]^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

COROLLARY

Following Corollaries can be derived from the theorem:

Corollary I: If $\xi(t) = t^{\alpha}$ then the degree of approximation of a function f belonging to the class Lip (α, p), $\frac{1}{p} < \alpha < 1$, is given by

$$\left\|t_n^{c_1,p_n}(x) - f(x)\right\|_p = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right).$$

Proof: Since $\left\|t_n^{c_1,p_n}(x) - f(x)\right\|_p = O\left((n+1)^{\frac{1}{p}}\ \frac{1}{(n+1)^{\alpha}}\right)$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right)$$

which complete the proof of corollary I.

Corollary II: If $p \to \infty$ in Corollary I, the degree of approximation of a function f belonging to the class Lip α, $0 < \alpha < 1$, is given by

$$\left\|t_n^{c_1,p_n}(x) - f(x)\right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right).$$

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