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SOME COMMON FIXED POINT THEOREMS IN MENGER SPACES USING OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS

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ABSTRACT
In this paper we prove some common fixed point theorems for family of occasionally weakly compatible mappings in Menger space. Also improvement of the results of B. D. Pant and Sunny Chauhan [1] under relaxed conditions is given.

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Keywords: Triangle function (t-norm); Menger spaces; Common fixed points; Compatible mappings; Weakly compatible mappings; Occasionally weakly compatible mappings.

INTRODUCTION
The concept of probabilistic metric space was first introduced and studied by Menger [2]. It is a probabilistic generalization of metric space in which we assign to any two points x and y, a distribution function $F_{x,y}$. The study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [3]. Fixed point theory is one of fruitful and effective tools in mathematics which has many applications within as well as outside mathematics. In this theory a contraction is one of the main tools to prove the existence and uniqueness results on fixed points in probabilistic analysis. Jungck [4] proved fixed point theorem for pair of commuting mappings in metric space as a generalization of the well known Banach contraction principle. Sessa [5] generalized this result by using the weaker hypothesis than commutativity called weak commutativity. The concept of weak commutativity in probabilistic settings was first studied by Singh and Pant [6, 7].

Jungck [8] introduced the notion of compatibility. This idea introduced by Mishra [9] in Menger space which has been weakened by appearing the concept of weak compatibility by Jungck and Rhoades [10].

Cho, Sharma and Sahu [11] introduced the concept of semi-compatibility mappings in a d-topological space. Singh and Jain [12] established some fixed point theorems in Menger space using semi-compatibility of the mappings. More recently, Al-Thagafi and Shahzad [13] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible (owc) mappings which is most general among all the commutativity concepts. The notion of occasionally weakly compatible (owc) mappings has become an area of interest for specialists in fixed point theory.

In this paper we obtain some common fixed point theorems for three pairs and family of self mappings under the condition of occasionally weak compatibility (owc) in Menger spaces, we improve and extend the results of Pant and Chauhan [1] and other results on compatible or weakly compatible mappings under relaxed conditions in Menger space.We first give some

preliminaries and definitions.

PRELIMINARIES AND DEFINITIONS

Definition 2.1 [3]: A real valued function f on the set of real numbers is called a distribution function if it is non-decreasing, left continuous with $\inf_{u\in R}\mathrm{f}(u)=0$ and $\sup_{u\in R}\mathrm{f}(u)=1$.
The Heaviside function H is a distribution function defined by

$$H(u)=\begin{cases}0, & u\le 0\\ 1, & u>0.\end{cases}$$

Definition 2.2 [3]: Let X be a non-empty set and let L denote the set of all distribution functions defined on X. An ordered pair (X,F) is called a probabilistic metric space where F is a mapping from $X\times X$ into L if for every pair $(x,y)\in X$ a distribution function $F_{x,y}$ assumed to satisfy the following conditions:
(1) $F_{x,y}(u)=H(u)$ if and only if $x=y$;
(2) $F_{x,y}(u)=F_{y,x}(u)$;
(3) $F_{x,y}(0)=0$;
(4) If $F_{x,y}(u_1)=1$ and $F_{y,z}(u_2)=1$, then $F_{x,z}(u_1+u_2)=1$ for all x,y,z in X and $u_1,u_2\ge 0$.
Definition 2.3 [3]: A t-norm is a function $t:[0,1]\times[0,1]\to[0,1]$ satisfying the following conditions:
(T1) $t(a,1)=a, t(0,0)=0$;
(T2) $t(a,b)=t(b,a)$;
(T3) $t(c,d)\ge t(a,b)$ for $c\ge a, d\ge b$;
(T4) $t(t(a,b),c)=t(a,t(b,c))$ for all a,b,c in $[0,1]$.
Definition 2.4 [3]: A Menger probabilistic metric space is an ordered triple (X,F,t), where t is a t-norm, and (X,F) is a probabilistic metric space satisfying the following condition:

$$F_{x,z}(u_1+u_2)\ge t(F_{x,y}(u_1),F_{y,z}(u_2)) \text{ for all } x,y,z \text{ in } X \text{ and } u_1,u_2\ge 0.$$

Definition 2.5 [9]: A sequence $\{x_n\}$ in (X,F,t) is said to converge to a point $x\in X$ if for every $\varepsilon>0$ and $\lambda>0$, there exists a positive integer $N(\varepsilon,\lambda)$ such that $F_{x_n,x}(\varepsilon)>1-\lambda$ for all $n\ge N(\varepsilon,\lambda)$.
Definition 2.6 [9]: A sequence $\{x_n\}$ in (X,F,t) is said to be a Cauchy sequence if for every $\varepsilon>0$ and $\lambda>0$, there exists a positive integer $N(\varepsilon,\lambda)$ such that $F_{x_n,x_m}(\varepsilon)>1-\lambda$ for all $n,m\ge N(\varepsilon,\lambda)$.
Definition 2.7 [9]: A Menger space (X,F,t) with continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X.
Definition 2.8: A coincidence point of two mappings is a point in their domain having the same image point under both mappings.

Formally, given two mappings $\mathrm{f},\mathrm{g}:X\to Y$ we say that a point x in X is a coincidence point of f and g if $f(x)=g(x)$.

Definition 2.9 [9]: Two self mappings A and B of a Menger space (X,F,t) are said to be compatible if $F_{ABx_n,BAx_n}(t)\to 1$ for all $t>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as $n\to\infty$.

Definition 2.10 [10]: Two self mappings A and B of a Menger space (X,F,t) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Ax = Bx$ for some $x\in X$, then $ABx = BAx$.

Remark 2.1 [14]: Two compatible self mappings are weakly compatible but the converse is not true. Therefore, the concept of weak compatibility is more general than that of compatibility.

Example 2.1: Let $X = R$ be the set of real numbers. Define $A,B: X\to X$ by

$$A(x)=\begin{cases}0, & x=-1,\\ x^2, & x\neq -1,\end{cases} \quad \text{and} \quad B(x)=2x-1, \forall x\in X.$$

For any sequence $\{x_n\}$ in X with $Ax_n, Bx_n \to z$ as $n\to\infty$ for some $z\in X$, we find that ABx_n, BAx_n converge to the same point in X, i.e., $\lim_{n\to\infty} F_{ABx_n,BAx_n}(t)=1$. For example , take $x_n=\{\frac{n}{n+1}\}$,we can find $Ax_n=(\frac{n}{n+1})^2$, $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}(\frac{n}{n+1})^2=\lim_{n\to\infty}(\frac{1}{1+\frac{1}{n}})^2=1$,

$Bx_n=\frac{2n}{n+1}-1=\frac{n-1}{n+1}$, $\lim_{n\to\infty}Bx_n=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{1}{n}}=1$. Also, $ABx_n=(\frac{n-1}{n+1})^2$,

$\lim_{n\to\infty}ABx_n=\lim_{n\to\infty}(\frac{1-\frac{1}{n}}{1+\frac{1}{n}})^2=1$, $BAx_n=2(\frac{n}{n+1})^2-1$, $\lim_{n\to\infty}BAx_n=2\lim_{n\to\infty}(\frac{n}{n+1})^2-1=1$.

Therefore, $\lim_{n\to\infty}F_{ABx_n,BAx_n}(t)=F_{1,1}(t)=1$. Hence, the pair (A,B) is compatible. Also, 1 is the unique coincidence point for A,B and $AB(1)=BA(1)$ then they are weakly compatible.

Definition 2.11 [13]: *Self mappings A and B of a Menger space (X,F,t) are said to be occasionally weakly compatible if and only if there exists a point $x\in X$ such that $Ax = Bx$ and $ABx = BAx$.*

Remark 2.2: *we can say that A and B are (owc) if there exists a point $x\in X$ such that $Ax = Bx$ and $AAx = BAx$.*

Definition 2.12 [12]: A pair (A,B) of self mappings of a Menger space (X,F,t) is said to be semi-compatible if $F_{ABx_n,Bx}(t)\to 1$ for all $t>0$,whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n\to x$ for some $x\in X$ as $n\to\infty$.

Remark 2.3: *In nontrivial case in which there is at least one coincidence point. the concept of occasionally weak compatibility is more general than that of weak compatibility.*

Example 2.2: *Let $X=[0,1]$. Define $A,B: X\to X$ by*

$$A(x)=\begin{cases}1-x, & 0\le x\le \frac{1}{2},\\ \frac{x}{2}, & \frac{1}{2}<x\le 1,\end{cases} \quad \text{and} \quad B(x)=\begin{cases}x, & 0\le x\le \frac{1}{2},\\ \frac{x}{2}, & \frac{1}{2}<x\le 1.\end{cases}$$

For any $x\in(\frac{1}{2},1]$, $ABx = A\frac{x}{2}=1-\frac{x}{2}$ and $BAx = B\frac{x}{2}=\frac{x}{2}$. Since, $1-\frac{x}{2}\neq\frac{x}{2}$ for all $x\in(\frac{1}{2},1]$ then A and B are not (wc). Moreover, at $x=\frac{1}{2}$, $A(\frac{1}{2})=B(\frac{1}{2})$ and $AB(\frac{1}{2})=BA(\frac{1}{2})$, implies, A and B are (owc).

Lemma 2.1 [9]: Let (X,F,t) be a Menger space. If there exists $k\in(0,1)$ such that:

$$F_{x,y}(kt)\ge F_{x,y}(t)$$

for all $x,y\in X$ and $t>0$ then $x=y$.

MAIN RESULTS

Pant and Chauhan [1] proved the following theorem.

Theorem 3.1: *Let A, B, S, T, L and M be self mappings on a complete Menger space (X,F,t) with continuous t-norm and t defined by $t(a,b)=\min\{a,b\}$ for all $a,b\in[0,1]$ and satisfy the following :*

(i) $AB(X)\subset M(X)$ and $ST(X)\subset L(X)$;

(ii) $M(X)$ and $L(X)$ are complete subspace of X;

(iii) Either AB or ST is continuous;

(iv) (AB,L) is semi-compatible and (ST,M) is weakly compatible;

(v) for all $p,q\in X$, $k\in(0,1)$ and $t>0$,

$$\begin{aligned}F^3_{ABp,STq}(kt)\ge\min\{&F^3_{Lp,Mq}(t),F^3_{ABp,Lp}(t),\\ &F^3_{STq,Mq}(t),F_{ABp,Mq}(2t),\\ &F_{STq,Lp}(2t),F^2_{STq,Mq}(t)\}.\end{aligned}$$

Then AB, ST, L and M have a unique common fixed point in X.

Motivated by the results of Pant and Chauhan [15] we improve theorem 3.1 .Our improvements are:

(1) Relaxed the continuity requirement of mappings completely.

(2) Completeness of the whole space.

(3) Weakened the concepts of semi-compatibility and weak compatibility by more general concept of occasionally weak compatibility.

(4) Minimal type contractive condition used.

Theorem 3.2: *Let A, B, S, T, L and M be self mappings on a Menger space (X,F,t) with continuous t-norm and t defined by $t(a,b)=\min\{a,b\}$ for all $a,b\in[0,1]$ and satisfy the following :*

(a) the pairs $\{AB,L\}$ and $\{ST,M\}$ are (owc);

(b) $AB=BA$, $ST=TS$, $LB=BL$ and $MT=TM$;

(c) for all $p,q\in X$, $k\in(0,1)$ and $t>0$,

$$F^3_{ABp,STq}(kt) \geq \min\{F^3_{Lp,Mq}(t), F^3_{ABp,Lp}(t),$$
$$F^3_{STq,Mq}(t), F_{ABp,Mq}(2t), F_{STq,Lp}(2t)\}.$$

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof: Since (AB, L) is occasionally weakly compatible then there exists a point x in X such that $ABx = Lx$ and $ABABx = LABx$. Also, as (ST, M) is occasionally weakly compatible then there
exists a point y in X such that $STy = My$ and $STSTy = MSTy$

Step 1: Putting $p = x$, $q = y$ in (c), we get:

$$F^3_{ABx,STy}(kt) \geq \min\{F^3_{Lx,My}(t), F^3_{ABx,Lx}(t),$$
$$F^3_{STy,My}(t), F_{ABx,My}(2t), F_{STy,Lx}(2t)\},$$
$$\geq \min\{F^3_{ABx,STy}(t), 1, 1,$$
$$F_{ABx,STy}(2t), F_{STy,ABx}(2t)\},$$
$$\geq F^3_{ABx,STy}(t).$$

By lemma 2.1, we have $ABx = STy$. Therefore, $Lx = ABx = STy = My = z$ Now, we show that z is a fixed point of AB.

Step 2: Putting $p = ABx$, $q = y$ in (c), we get:

$$F^3_{ABABx,STy}(kt) \geq \min\{F^3_{LABx,My}(t), F^3_{ABABx,LABx}(t),$$
$$F^3_{STy,My}(t), F_{ABABx,My}(2t), F_{STy,LABx}(2t)\},$$
$$\geq \min\{F^3_{ABABx,STy}(t), 1, 1,$$
$$F_{ABABx,STy}(2t), F_{STy,ABABx}(2t)\},$$
$$\geq F^3_{ABABx,STy}(t).$$

Thus, $F_{ABABx,STy}(kt) \geq F_{ABABx,STy}(t)$, for all $t > 0$. $\Rightarrow ABABx = STy$. Therefore,

$$z = Lz = ABz. \qquad (3.1)$$

Step 3: Putting $p = Bz$, $q = y$ in (c), we get:

$$F^3_{ABBz,STy}(kt) \geq \min\{F^3_{LBz,My}(t), F^3_{ABBz,LBz}(t),$$
$$F^3_{STy,My}(t), F_{ABBz,My}(2t), F_{STy,LBz}(2t)\}.$$

As $LB = BL$ and $AB = BA$, so $L(Bz) = B(Lz) = Bz$ and $ABBz = B(ABz) = Bz$. Hence,

$$F^3_{Bz,z}(kt) \geq F^3_{Bz,z}(t) \qquad forall \qquad t > 0$$

Since $ABz = z$ and $Bz = z$, then $Az = z$. Thus,

$$z = Lz = Az = Bz. \qquad (3.2)$$

Similarly, using the fact that the pair (ST, M) is (owc) then $STz = STSTy = MSTy = Mz$. Now, we show that z is a fixed point of ST.

Step 4: Putting $p = z$, $q = STy$ in (c), we get:

$$F^3_{ABz,STSTy}(kt) \geq \min\{F^3_{Lz,MSTy}(t), F^3_{ABz,Lz}(t),$$
$$F^3_{STSTy,MSTy}(t), F_{ABz,MSTy}(2t), F_{STSTy,Lz}(2t)\},$$

$$F^3_{z,STz} \geq \min\{F^3_{z,STz}(t),1,1,F_{z,STz}(2t),F_{STz,z}(2t)\},$$
$$\geq F^3_{z,STz}(t).$$

Thus, $F_{z,STz}(kt) \geq F_{z,STz}(t)$, for all $t > 0$. $\Rightarrow z = STz$. Therefore,

$$z = Mz = STz. \quad (3.3)$$

Step 5: Putting $p = z$, $q = Tz$ in (c), we get:

$$F^3_{ABz,STTz}(kt) \geq \min\{F^3_{Lz,MTz}(t),F^3_{ABz,Lz}(t),$$
$$F^3_{STTz,MTz}(t),F_{ABz,MTz}(2t),F_{STTzy,Lz}(2t)\}.$$

As $MT = TM$ and $ST = TS$, so $M(Tz) = T(Mz) = Tz$ and $STTz = T(STz) = Tz$. Hence, $F^3_{z,Tz}(kt) \geq F^3_{z,Tz}(t)$ $forall$ $t > 0$,

Since $STz = z$ and $Tz = z$, then $Sz = z$. Thus,

$$z = Mz = Sz = Tz. \quad (3.4)$$

Step 6(uniqueness): Let w be another fixed point of the six mappings, then $Aw = Bw = Lw = Mw = Tw = Sw = w$.

Putting $p = z$ and $q = w$ in (c), to obtain:

$$F^3_{ABz,STw}(kt) \geq \min\{F^3_{Lz,Mw}(t),F^3_{ABz,Lz}(t),$$
$$F^3_{STw,Mw}(t),F_{ABz,Mw}(2t),F_{STw,Lz}(2t)\},$$
$$F^3_{z,w}(kt) \geq \min\{F^3_{z,w}(t),F^3_{z,z}(t),F^3_{w,w}(t),$$
$$F_{z,w}(2t),F_{w,z}(2t)\},$$
$$\geq \min\{F^3_{z,w}(t),1,1,F_{z,w}(2t),$$
$$F_{w,z}(2t)\},$$
$$\geq F^3_{z,w}(t).$$

Thus, $F_{z,w}(kt) \geq F_{z,w}(t)$ for all $t > 0$ .then $z = w$ and z is a unique common fixed point of the six mappings.

As a generalization of theorem 3.2, we prove the following result for a finite family of self mappings.

Theorem 3.3: Let P_1, $P_2,...,P_n,\ldots,P_{2n}$ be self mappings on a Menger space (X,F,t) with continuous t-norm , $t(a,b) = min\{a,b\}$ for all $a,b \in [0,1]$ and satisfy the following :

(a) the pairs $\{P_3P_5...P_{2n-1},P_1\}$ and $\{P_4P_6...P_{2n},P_2\}$ are (owc);

(b) $P_3(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_3$,

$P_3P_5(P_7...P_{2n-1}) = (P_7...P_{2n-1})P_3P_5$,

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$P_3P_5...P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_3P_5...P_{2n-3}$,

$P_1(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_1$,

$P_1(P_7...P_{2n-1}) = (P_7...P_{2n-1})P_1$,

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$P_1P_{2n-1} = P_{2n-1}P_1,$

$P_4(P_6...P_{2n}) = (P_6...P_{2n})P_4,$

$P_4P_6(P_8...P_{2n}) = (P_8...P_{2n})P_4P_6,$

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$P_4P_6...P_{2n-2}(P_{2n}) = (P_{2n})P_4P_6...P_{2n},$

$P_2(P_6...P_{2n}) = (P_6...P_{2n})P_2,$

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$P_2P_{2n} = P_{2n}P_2,$

(c) for all $p, q \in X$, $k \in (0,1)$ and $t > 0$,

$$F^3_{P_3P_5...P_{2n-1}p, P_4P_6...P_{2n}q}(kt) \geq \min\{F^3_{P_1p, P_2q}(t), F^3_{P_3P_5...P_{2n-1}p, P_1p}(t), F^3_{P_4P_6...P_{2n}q, P_2q}(t), F_{P_3P_5...P_{2n-1}p, P_2q}(2t), F_{P_4P_6...P_{2n}q, P_1p}(2t)\}.$$

Then $P_1,...P_{2n}$ have a unique common fixed point in X.

Proof: Since $(P_3...P_{2n-1}, P_1)$ is occasionally weakly compatible then there exists a point x in X such that: $P_3...P_{2n-1}x = P_1x$ *and* $P_3...P_{2n-1}P_3...P_{2n-1}x = P_1P_3P_5...P_{2n-1}x$. Also, as $(P_4...P_{2n}, P_2)$ is occasionally weakly compatible then there exists a point y in X such that:

$P_4...P_{2n}y = P_2y$ and $P_4...P_{2n}P_4...P_{2n}y = P_2P_4...P_{2n}y$.

Step 1: Putting $p = x$, $q = y$ in (c), we get:

$$\begin{aligned} F^3_{P_3...P_{2n-1}x, P_4...P_{2n}y}(kt) &\geq \min\{F^3_{P_1x, P_2y}(t), F^3_{P_3...P_{2n-1}x, P_1x}(t), \\ &\quad F^3_{P_4...P_{2n}y, P_2y}(t), F_{P_3...P_{2n-1}x, P_2y}(2t), F_{P_4...P_{2n}y, P_1x}(2t)\}, \\ &\geq \min\{F^3_{P_3...P_{2n-1}x, P_4...P_{2n}y}(t), 1, 1, \\ &\quad F_{P_3...P_{2n-1}x, P_4...P_{2n}y}(2t), F_{P_4...P_{2n}y, P_3...P_{2n-1}x}(2t)\}, \\ &\geq F^3_{P_3...P_{2n-1}x, P_4...P_{2n}y}(t). \end{aligned}$$

Hence by lemma (2.1), we have $P_3...P_{2n-1}x = P_4...P_{2n}y$. Therefore, $P_1x = P_3...P_{2n-1}x = P_4...P_{2n}y = P_2y = z$ Now, we show that z is a fixed point of $P_3...P_{2n-1}$.

Step 2. Putting $p = P_3...P_{2n-1}x$, $q = y$ in (c), we get:

$$F^3_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(kt) \geq \min\{F^3_{P_1P_3...P_{2n-1}{}^x,P_2{}^y}(t),$$
$$F^3_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_1P_3...P_{2n-1}{}^x}(t), F^3_{P_4...P_{2n}{}^y,P_2{}^y}(t),$$
$$F_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_2{}^y}(2t), F_{P_4...P_{2n}{}^y,P_1P_3...P_{2n-1}{}^x}(2t)\},$$
$$\geq \min\{F^3_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(t),1,1,$$
$$F_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(2t), F_{P_4...P_{2n}{}^y,P_3...P_{2n-1}P_3...P_{2n-1}{}^x}(2t)\},$$
$$\geq F^3_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(t).$$

Thus, $F_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(kt) \geq F_{P_3...P_{2n-1}P_3...P_{2n-1}{}^x,P_4...P_{2n}{}^y}(t)$,

for all $t > 0 \Rightarrow P_3...P_{2n-1}P_3...P_{2n-1}x = P_4...P_{2n}y$. Therefore,

$$z = P_1z = P_3...P_{2n-1}z. \qquad (3.5)$$

Step 3: Putting $p = P_5...P_{2n-1}z$, $q = y$ in (c), we get:

$$F^3_{P_3...P_{2n-1}P_5...P_{2n-1}{}^z,P_4...P_{2n}{}^y}(kt) \geq \min\{F^3_{P_1P_5...P_{2n-1}{}^z,P_2{}^y}(t),$$
$$F^3_{P_3...P_{2n-1}P_5...P_{2n-1}{}^z,P_1P_5...P_{2n-1}{}^z}(t), F^3_{P_4...P_{2n}{}^y,P_2{}^y}(t),$$
$$F_{P_3...P_{2n-1}P_5...P_{2n-1}{}^z,P_2{}^y}(2t), F_{P_4...P_{2n}{}^y,P_1P_5...P_{2n-1}{}^z}(2t)\}.$$

As $P_1(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_1$ and $P_3(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_3$,

so $P_1(P_5...P_{2n-1})z = (P_5...P_{2n-1})P_1z = P_5...P_{2n-1}z$

and $P_3P_5...P_{2n-1}P_5...P_{2n-1}z = P_5...P_{2n-1}(P_3...P_{2n-1}z) = P_5...P_{2n-1}z$. Hence,

$$F^3_{P_5...P_{2n-1}{}^z,z}(kt) \geq F^3_{P_5...P_{2n-1}{}^z,z}(t) \text{ for all } t > 0.$$

Since $P_3P_5...P_{2n-1}z = z$ and $P_5...P_{2n-1}z = z$, then $P_3z = z$. Thus, $z = P_1z = P_3z = P_5...P_{2n-1}z$.

Continuing this procedure, we obtain

$$z = P_1z = P_3z = P_5z = ... = P_{2n-1}z. \qquad (3.6)$$

Similarly, using the fact that $P_4...P_{2n}z = P_4...P_{2n}P_4...P_{2n}y = P_2P_4...P_{2n}y = P_2z$. Now, we show that z is a fixed point of $P_4...P_{2n}$.

Step 4: Putting $p = z$, $q = P_4...P_{2n}y$ in (c), we get:

$$F^3_{P_3...P_{2n-1}{}^z,P_4...P_{2n}P_4...P_{2n}{}^y}(kt) \geq \min\{F^3_{P_1{}^z,P_2P_4...P_{2n}{}^y}(t),$$
$$F^3_{P_3...P_{2n-1}{}^z,P_1{}^z}(t), F^3_{P_4...P_{2n}P_4...P_{2n}{}^y,P_2P_4...P_{2n}{}^y}(t),$$
$$F_{P_3...P_{2n-1}{}^z,P_2P_4...P_{2n}{}^y}(2t), F_{P_4...P_{2n}P_4...P_{2n}{}^y,P_1{}^z}(2t)\},$$
$$F^3_{z,P_4...P_{2n}{}^z} \geq \min\{F^3_{z,P_4...P_{2n}{}^z}(t),1,1,$$
$$F_{z,P_4...P_{2n}{}^z}(2t), F_{P_4...P_{2n}{}^z,z}(2t)\},$$
$$\geq F^3_{z,P_4...P_{2n}{}^z}(t).$$

Thus, $F_{z,P_4...P_{2n}{}^z}(kt) \geq F_{z,P_4...P_{2n}{}^z}(t)$, for all $t > 0$. $\Rightarrow z = P_4...P_{2n}z$. Therefore,

$$z = P_2z = P_4...P_{2n}z. \qquad (3.7)$$

Step 5: Putting $p = z$, $q = P_6...P_{2n}z$ in (c), we get:

$$F^3_{P_3...P_{2n-1}z, P_4...P_{2n}P_6...P_{2n}z}(kt) \geq \min\{F^3_{P_1z, P_2P_6...P_{2n}z}(t),$$
$$F^3_{P_3...P_{2n-1}z, P_1z}(t), F^3_{P_4...P_{2n}P_6...P_{2n}z, P_2P_6...P_{2n}z}(t), \quad \text{As } P_2(P_6...P_{2n}) = (P_6...P_{2n})P_2$$
$$F_{P_3...P_{2n-1}z, P_2P_6...P_{2n}z}(2t), F_{P_4...P_{2n}P_6...P_{2n}z, P_1z}(2t)\}.$$

and $P_4(P_6...P_{2n}) = (P_6...P_{2n})P_4$,

so $P_2(P_6...P_{2n}z) = P_6...P_{2n}(P_2z) = P_6...P_{2n}z$

and $P_4...P_{2n}P_6...P_{2n}z = P_6...P_{2n}(P_4...P_{2n}z) = P_6...P_{2n}z$. Hence,

$F^3_{z, P_6...P_{2n}z}(kt) \geq F^3_{z, P_6...P_{2n}z}(t)$ *for all* $t > 0$.

Since $P_4...P_{2n}z = z$ and $P_6...P_{2n}z = z$, then $P_4z = z$. Thus, $z = P_2z = P_4z = P_6...P_{2n}z$.

Continuing this procedure, we obtain

$$z = P_2z = P_4z = P_6 = ... = P_{2n}z. \qquad (3.8)$$

Step 6(uniqueness): Let w be another fixed point of these family, then $P_1w = P_2w = ... = P_nw = w$.

Putting $P = z$ and $q = w$ in (c), to obtain:

$$F^3_{P_3...P_{2n-1}z, P_4...P_{2n}w}(kt) \geq \min\{F^3_{P_1z, P_2w}(t), F^3_{P_3...P_{2n-1}z, P_1z}(t),$$
$$F^3_{P_4...P_{2n}w, P_2w}(t), F_{P_3...P_{2n-1}z, P_2w}(2t), F_{P_4...P_{2n}w, P_1z}(2t)\},$$

$$F^3_{z,w}(kt) \geq \min\{F^3_{z,w}(t), F^3_{z,z}(t), F^3_{w,w}(t),$$
$$F_{z,w}(2t), F_{w,z}(2t)\},$$
$$\geq F^3_{z,w}(t).$$

Thus, $F_{z,w}(kt) \geq F_{z,w}(t)$ for all $t > 0$. then $z = w$ and z is a unique common fixed point of the family of mappings.

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