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A STUDY ON ESSENTIAL SUBMODULE AND SINGULAR MODULE

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ABSTRACT
In this paper we discus about direct sum, essential submodule, singular module, essential monomorphism and some theorems.

Key Words: ***Ring, Module, Direct sum, Essential submodule, Singular module.***

1. INTRODUCTION
A ring is a system(R, +, ., 0, 1) consisting of a set R, two binary operations, addition(+) and multiplication(.) and two elements $0\neq 1$ of R such that (R,+, 0) is an abelian group, (R, ., 1) is a semi group with identity and multiplication is both left and right distributive over addition. A ring whose multiplicative structure is commutative is called a commutative ring. A subset I of a ring R is a two sided ideal of R in case it is an additive subgroup such that for all $x \in I$ and all a, $b \in R$

$$axb \in I$$

The two subsets {0} and R both ideals of R ; these are called the trivial ideals of R. Any ideal of R other than R itself is called a proper ideal. The ideal{0} ,which we frequently denote simply by 0, is called the zero ideal.

The ring R is simple if {0 } and R are the only ideals of R. A ring R with identity $1\neq 0$ in which every non zero element a is unit, is called a division ring. Thus every division ring is a simple ring, for let R be a division ring and I be an ideal of R such that $I \neq \{0\}$, then there exists at least one $a \in I$ such that $a \neq 0$. Since R is a division ring, so $a^{-1} \in R$ and $aa^{-1} = 1$. since $a \in I$, $a^{-1} \in R$, $aa^{-1} \in I$, by definition of an ideal , or $1 \in I$,therefore I = R since if I is an ideal of a ring R with unity such that $1 \in I$, then I= R.

On the other hand every commutative simple ring is a field but in general simple rings need not be division ring, and division rings need not be commutative, for example, let H be the subset of $M_2(\mathbf{C})$ the 2 x 2 matrices over the complex field, of all elements of the form

$$q=\begin{pmatrix} a+ib & c+id \\ c-id & a-ib \end{pmatrix}$$

with a,b, c,d ∈ R,then H is a subring of ,$M_2(\mathbf{C})$. Consider the elements

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ i=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},k=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in H.$$

Thus above typical element q of H is q = a1+ bi+ cj+dk, then if $q\neq 0$, it is invertible ,that is H is a division ring but it is not commutative.

Let R be a ring(with or without 1 and commutative or not). By a left R- module M, we mean, an abelian group (M, +) together with a map $R \times M \to M$, $ax \mapsto ax$, called the scalar multiplication or the structure map, such that

1. $r(x+y)= rx+ry$ for all $r \in R$ and $x, y \in M$
2. $(r_1+r_2)x = r_1x + r_2x$ for all $r_1,r_2 \in R$ and $x \in M$ and
3. $(r_1r_2)x= r_1(r_2x)$ for all $r_1, r_2 \in R$ and $x \in M$.

Elements of R are called scalars. The left R-Module is denoted by R^M .
Let M be an R- module. A nonempty subset N of M is called an R-submodule of M if N is an additive subgroup of M i.e. $x, y \in N$ implies $x-y \in N$ and N is closed for scalar multiplication i.e. $x \in N$, $a \in R$ implies $ax \in N$.

Let M be an R-module. A submodule K of a left R- module R^M is called a direct summand of M if there exists a submodule K' of M such that $M = K \oplus K'$, that is $M= K+K'$ and $K \cap K'= 0$.
A submodule K of R^M is called essential (or large) submodule in M if whenever L is a submodule of M such that $K \cap L = 0$ implies $L = 0$ and it is denoted by $K \,\Delta\, M$ and read as " K is essential in M."
For example, if M is a left R module. Then $M \,\Delta\, M$ because if $M \cap L =0$ implies $L= 0$, since $L \leq M$.

The two concepts, direct summand and essential submodule are reminiscent of the topological concepts of connected components and dense.

2 MAIN RESULTS

Lemma 1: Let L (R) be the set of all essential left ideals of R. Let $I \in L(R)$ and $r \in R$ .Then

$Ir^{-1}= \{a \in R: ar \in I \}$

is an essential left ideal of R. i.e. $Ir^{-1} \in L(R)$.

Proof: Let $Ir^{-1} \cap K =0$ where K is a left ideal of R. Then $I \cap Kr= 0$ where Kr is a left ideal of R.

or $Kr = 0 \subseteq I$, since $I \,\Delta\, R$. So $Kr \subseteq I$ implies $K \subseteq Ir^{-1}$.Therefore $K = Ir^{-1} \cap K = 0$ implies $K= 0$. Hence Ir^{-1} is an essential left ideal of R.

Lemma 2: Let A be a left R- module. Then

$Z(A)= \{ x \in A : Ix= 0 \text{ for some } I \in L(R)\}$

is a submodule of A.

Proof: (i) Since $R \in L(R)$ implies $R0 = 0$ implies $0 \in Z(A)$.

(ii) If $x, y \in Z(A)$ then $Ix = 0$, $Jy=0$ for some $I,J \in L(R)$. Since $(I\cap J) \in L(R)$ .Therefore

$(I\cap J)(x-y)= 0$ implies $x-y \in Z(A)$.

(iii) Given any $r \in R$, we have $Ir^{-1} \in L(R)$ where $Ir^{-1}= \{a \in R: ar \in I \}$. So $(Ir^{-1})(rx) \leq Ix =0$ since $I \in L(R)$. Therefore $(Ir^{-1})(rx)=0$. So $rx \in Z(A)$.

Hence Z(A) is a submodule of A.

Definition: A pair of module homomorphism, $A \xrightarrow{f} B \xrightarrow{g} C$, is said to be exact at B if image of f = kernel of g. An exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short exact sequence; note that f is a monomorphism and g is an epimorphism. A monomorphism f: $R^M \to R^N$ is called essential monomorphism if Im f Δ N.

Theorem 1: A module C is singular if and only if there exists a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that f is an essential monomorphism.

Proof: Assume we have such exact sequence. Let b $\in$ B. Define a map φ from R to B such that K be a left ideal of R such that I$\cap$ K= 0. Then f(A) $\cap$ Kb= 0, where Kb is a submodule of B which implies that Kb= 0, since f(A) Δ B and Kb $\subseteq$ f(A) and K$\subseteq$ I. So K= K $\cap$ I= 0. Therefore I= φ^{-1}(f(A)) $\in$ L (R). Now Ib $\subseteq$ f(A) = ker g, by exact sequence which implies that g(Ib)= 0 or I(g(b))= 0. Then g(b) $\in$ Z(C). Therefore C= g(B) $\subseteq$ Z(C), since g is onto. Hence C= Z(C) and C is singular.

Conversely, let C is singular. Then Z(C) = C. We choose a short exact sequence
$0 \to A \xrightarrow{\subseteq} B \xrightarrow{g} C \to 0$ such that B is free. If $\{b_\alpha\}_\alpha$ is a basis of B. Then for each α, we have $I_\alpha g(b_\alpha)$= 0 for some $I_\alpha \in$ L (R). Therefore $g(I_\alpha b_\alpha)$= 0. $I_\alpha b_\alpha \subseteq$ ker g= im i = A. Since I_α is essential in R^R, Then $I_\alpha b_\alpha$ is also essential in Rb_α for all α, for if $I_\alpha b_\alpha \cap K$= 0, then $I_\alpha \cap Kb^{-1}_\alpha$ = 0. Therefore Kb^{-1}_α= 0, since $I_\alpha \ \Delta$ R and $Kb^{-1}_\alpha = \{r \in R: rb_\alpha \in K\}$ which implies K= 0. So $I_\alpha b_\alpha \ \Delta \ Rb_\alpha$ implies $\oplus I_\alpha b_\alpha \ \Delta \ Rb_\alpha$= B. Then A Δ B, since $\oplus I_\alpha b_\alpha \subseteq A \subseteq B$. Hence i: A→ B is an essential monomorphism.

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