

COINCIDENCE AND COMMON FIXED POINT THEOREM IN CONE METRIC

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ABSTRACT The aim of this paper is to establish a coincidence and common fixed point theorem for four self maps satisfying a contractive condition in a cone metric space. The intent of this paper is to introduce the concept of compatibility without assuming its normality.

Keywords: Weakly compatible maps, coincidence point, cone metric space, common fixed point.

SUBJECT CLASSIFICATION CODES: 47H10, 54H25

1. INTRODUCTION

Cone metric spaces were introduced by Huang and Zhang in [4], replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. They introduced the basic definitions and discuss some properties of convergence of sequences in cone metric spaces. They also obtained various fixed point theorems for contractive single - valued maps in such spaces. They have proved some fixed point theorems of contractive mappings on cone metric spaces.

In Abbas and Jungck [1] generalized the result of [4] for two self maps through weak compatibility in a normal cone metric space. On the same line Vetro [13] proved some fixed point theorem for two self maps satisfying a contractive condition through weak compatibility. The authors [6] introduced to concept of compatibility and proved fixed point theorems for contractive type mappings in cone metric space [2, 5, 11]. Since the Banach Contraction Principle, several types of generalization contraction of mappings on metric spaces have appeared. Afterwards, some authors have proved some common fixed point theorems with normal and non-normal cones in these spaces such as [2, 5, 9, 10]. Jungck [10] introduced the concept of weakly compatible maps.

Recently, Rezapour and Hambarani [11] were able to omit the assumption of normality in a cone metric space, which is a milestone in developing fixed point theory. Also in [3] Arshad et al. proved a fixed point theorem for three self map adopting the contractive condition of [12] through weak compatibility. The aim of this paper is to prove a common fixed point theorem four for self mappings through weak compatibility in cone metric space. Our results generalize [7] and [8] extend and unify several well-known fixed point results in cone metric spaces.

2. PRELIMINARIES

DEFINITION 2.1 Let E be a real Banach space and P be a subset of E . P is called cone if,

- (i) P is closed, non-empty and $P \neq 0$;
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;
- (iii) $x \in P$ and $-x \in P$ imply $x = 0$;

Given a cone $P \subseteq E$, we define a partial ordering " \leq " in E by $x \leq y$ if $y - x \in P$. we write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^0$, where P^0 stands for some interior of P .

Note 1. For $a \geq 0$ and $x \in P$, taking $b = 0$ in (ii) we have $ax \in P$.

2. taking $a = b = 0$ in (ii) we have $0 \in P$.

PROPOSITION 2.1 Let P be a cone in a real Banach space E . If $a \in P$ and $a \leq ka$ for some $k \in [0, 1)$, then $a = 0$.

Proof: For $a \in P$, $k \in [0, 1)$ and $a \leq ka$ gives $(k - 1)a \in P$. As $0 \leq k < 1$ we have $(1 - k) > 0$ which gives $1/(1 - k) > 0$. So by Note 1, $\frac{1}{(1 - k)}(k - 1)a \in P$ implies $-a \in P$.

Now $a \in P$ and $-a \in P$, which implies $a = 0$, by Definition 1, (iii).

PROPOSITION 2.2 Let P be a cone in a real Banach space E . If $a \in E$ and $a \ll c$ for all $c \in P^0$, then $a = 0$.

DEFINITION 2.2 Let X be non-empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies:

- (a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$;

Then d is called a cone metric on X and (X, d) is called a cone metric space.

DEFINITION 2.3 Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \in c$ there is a positive integer N_c such that for all $n > N_c$ we have $d(x_n, x) \in c$, then the sequence $\{x_n\}$ is said to converge to x , and x is called the limit of $\{x_n\}$. We write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$, as $n \rightarrow \infty$.

DEFINITION 2.4 Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X . If for any $c \in E$ with $0 \ll c$ there is a N such that for all $n, m > N$ we have $d(x_n, x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X .

REMARK 2.1 We have $\lambda P^0 \subseteq P^0$ for $\lambda > 0$, and $P^0 + P^0 \subseteq P^0$.

DEFINITION 2.5 Let (X, d) be cone metric space. If every Cauchy sequence in X is convergent in X , then X is called complete cone metric space.

PROPOSITION 2.3 Let (X, d) be a cone metric space and P be a cone in a real Banach space E . If $u \leq v$, $v \ll w$ then $u \ll w$.

DEFINITION 2.6 Let f and g be self-maps on a set X . If $fx = gx$, for some $x \in X$ then x is called coincidence point of f and g .

DEFINITION 2.7 Let f and g be self-maps on a set X then f and g are said to be weakly compatible if they commute at coincidence points, i.e., $fu = gu$ for some $u \in X$ then $fgu = gfu$.

LEMMA 2.1 Let f and g be weakly compatible self maps of a set X , if f and g have a unique point coincidence $w = fx = gx$, then w , is the unique common fixed point of f and g .

LEMMA 2.2 Let (X, d) be a cone metric space with respect to a cone P in a real

Banach space E , and take $k_1, k_2, k > 0$. Suppose that $x_n \rightarrow x, y_n \rightarrow y$, and $z_n \rightarrow z$ in X and $ka \leq k_1 d(x_n, x) + k_2 d(y_n, y) + d(z_n, z)$. Then $a = 0$.

Proof: As $x_n \rightarrow x$ and $y_n \rightarrow y$, there exists a positive integer N_c such that

$$\frac{c}{(k_1 + k_2 + k_3)} - d(x_n, x), \frac{c}{(k_1 + k_2 + k_3)} - d(y_n, y), \frac{c}{(k_1 + k_2 + k_3)} - d(z_n, z) \in P^0 \text{ for all } n > N_c.$$

Therefore, by Remark 1, we have

$$\frac{k_1 c}{(k_1 + k_2 + k_3)} - k_1 d(x_n, x), \frac{c k_2}{(k_1 + k_2 + k_3)} - k_2 d(y_n, y), \frac{c k_3}{(k_1 + k_2 + k_3)} - k_3 d(z_n, z) \in P^0 \text{ for all } n > N_c$$

Again by adding and Remark 2.1, we have

$$c - k_1 d(x_n, x) - k_2 d(y_n, y) - k_3 d(z_n, z) \in P^0 \text{ for all } n > N_c.$$

From (1) and Proposition 2.3 we have $c - ka \in P^0$, i.e. $ka \ll c$ for each $c \in P^0$. Finally, by Proposition 2.2 we have $a = 0$ since $k > 0$.

3. RESULTS

Theorem 3.1 Let (X, d) be a cone metric space. Suppose mapping S, T, I and $J: X \rightarrow X$ satisfy

- 3.1.1 If $S(X) \subseteq J(X), T(X) \subseteq I(X)$;
- 3.1.2 One of $S(X), J(X), T(X)$ and $I(X)$ is complete subspace of X ;
- 3.1.3 The maps $\{S, I\}$ and $\{T, J\}$ have unique coincidence point in X ;
- 3.1.4 $\{T, J\}$ and $\{S, I\}$ are weakly compatible.
- 3.1.5 $d(Sx, Ty) \leq a_1 d(Ix, Jy) + a_2 d(Ix, Sx) + a_3 d(Jy, Ty) + a_4 d(Ix, Ty) + a_5 d(Jy, Sx)$ for all $x, y \in X$. where $a_i \geq 0$ ($i = 1, 2, 3, 4, 5$) $\in [0, 1)$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

Then the mappings S, T, I and J have a unique common fixed point in X .

Proof: Suppose x_0 be any arbitrary point in X then by (3.1.1) there exists $x_1, x_2 \in X$ such that

$$Sx_0 = Jx_1, Tx_1 = Ix_2$$

Inductively, we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Jx_{2n+1},$$

$$y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}; \quad n = 1, 2, 3, \dots$$

Firstly, we show that $\{y_n\}$ is a cauchy sequence in X .

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5) we get,

$$d(Sx_{2n}, Tx_{2n+1}) = d(y_{2n}, y_{2n+1})$$

$$\leq a_1 d(Ix_{2n}, Jx_{2n+1}) + a_2 d(Ix_{2n}, Sx_{2n}) + a_3 d(Jx_{2n+1}, Tx_{2n+1}) + a_4 d(Ix_{2n}, Tx_{2n+1})$$

$$+ a_5 d(Jx_{2n+1}, Sx_{2n})$$

$$= a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n-1}, y_{2n+1}) + a_5 d(y_{2n}, y_{2n})$$

$$= a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n}, y_{2n+1}) + a_4 [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

write $d(y_n, y_{n+1}) = d_n$, we have

$$d_{2n} \leq a_1 d_{2n-1} + a_2 d_{2n-1} + a_3 d_{2n} + a_4 [d_{2n-1} + d_{2n}]$$

$$d_{2n} (1 - a_3 - a_4) \leq (a_1 + a_2 + a_4) d_{2n-1}$$

$$d_{2n} \leq \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)} d_{2n-1}$$

$$\text{which implies } d_{2n} \leq k d_{2n-1} \tag{3.1}$$

$$\text{where } k = \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)} \text{ by (f), } k < 1.$$

Put $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.1.5) we get,

$$d(Sx_{2n+2}, Tx_{2n+1}) = d(y_{2n+2}, y_{2n+1})$$

$$\leq a_1 d(Ix_{2n+2}, Jx_{2n+1}) + a_2 d(Ix_{2n+2}, Sx_{2n+2}) + a_3 d(Jx_{2n+1}, Tx_{2n+1}) + a_4 d(Ix_{2n+2}, Tx_{2n+1}) + a_5 d(Jx_{2n+1}, Sx_{2n+2})$$

$$= a_1 d(y_{2n+1}, y_{2n}) + a_2 d(y_{2n+1}, y_{2n+2}) + a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n+1}, y_{2n+1}) + a_5 d(y_{2n}, y_{2n+2})$$

$$= a_1 d(y_{2n+1}, y_{2n}) + a_2 d(y_{2n+1}, y_{2n+2}) + a_3 d(y_{2n}, y_{2n+1}) + a_5 [d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n+2})]$$

$$\text{i.e., } d_{2n+1} \leq a_1 d_{2n} + a_2 d_{2n+1} + a_3 d_{2n} + a_5 [d_{2n} + d_{2n+1}]$$

$$d_{2n+1} (1 - a_2 - a_5) \leq (a_1 + a_3 + a_5) d_{2n}$$

$$d_{2n+1} \leq \frac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)} d_{2n}$$

$$\text{which implies } d_{2n+1} \leq h d_{2n} \tag{3.2}$$

$$\text{where } h = \frac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)}$$

In view (3.1) and (3.2), we have

$$d_{2n+1} \leq h d_{2n} \leq k h d_{2n-1} \leq k h^2 d_{2n-2} \leq \dots \leq k^n h^{n+1} d_0,$$

where $d_0 = d(y_0, y_1)$ and

$$d_{2n} \leq k d_{2n-1} \leq h k d_{2n-2} \leq h k^2 d_{2n-3} \leq \dots \leq h^n k^n d_0, \text{ where } d_0 = d(y_0, y_1)$$

Therefore $d_{2n+1} \leq k^n h^{n+1} d_0$, and $d_{2n} \leq h^n k^n d_0$. Also,

$$d(y_{n+p}, y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{n+p}, y_{n+p-1}),$$

$$\text{i.e., } d(y_{n+p}, y_n) \leq d_n + d_{n+1} + \dots + d_{n+p-1}.$$

(3.3)

we next suppose that $n = 2m$ is even, then by (3.3) we have

$$d(y_{2m+p}, y_{2m}) \leq [k^m h^m + h^{m+1} k^m + h^{m+1} k^{m+1} + h^{m+2} k^{m+1} \dots +] d_0.$$

$$= k^m h^m [1 + h + h k + h^2 k + h^2 k^2 + \dots +] d_0.$$

$$= k^m h^m [(1 + kh + k^2 h^2 + \dots) + (h + kh^2 + k^2 h^3 + \dots)] d_0.$$

$$= k^m h^m (1 + kh + k^2h^2 + \dots) (1 + h) d_0.$$

Since $kh < 1$ and P is closed, we conclude that

$$\text{Thus } d(y_{2m+p}, y_{2m}) \leq (kh)^m \frac{(1+h)}{(1-kh)} d_0. \quad (3.4)$$

Now for $c \in P^0$, there exists $r > 0$ such that $c - y \in P^0$ if $\|y\| < r$.

choose a positive integer N_c such that $\|(kh)^m (1 + h) d_0 / (1 - kh)\| < r$, for all $m \geq N_c$, which implies that

$$c - (kh)^m \frac{(1+h)}{(1-kh)} d_0 \in P^0. \text{ and } (kh)^m \frac{(1+h)}{(1-kh)} d_0 - d(y_{n+p}, y_n) \in P.$$

So we have $c - d(y_{2m+p}, y_{2m}) \in P^0$, for all $m > N_c$ and for all p by proposition 2.3.

The same case is true if $n = 2m + 1$ is odd. This implies $d(y_{n+p}, y_n) \ll c$, for all p and $n \geq N_c$, for all p . Hence $\{y_n\}$ is a Cauchy sequence in X .

Case I: Suppose $I(X)$ is complete. Since $\{y_n\}$ is a Cauchy sequence in X which is complete. So $y_{2n+1} \rightarrow z = Iu$ for some $u \in X$. Now

$$\begin{aligned} d(Su, Iu) &\leq d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, Iu) \\ &= d(Su, Tx_{2n+1}) + d(y_{2n+1}, Iu) \end{aligned}$$

Put $x = u$ and $y = x_{2n+1}$ in (3.1.5) we get,

$$\begin{aligned} d(Su, Iu) &= d(Su, Tx_{2n+1}) + d(y_{2n+1}, Iu) \\ &\leq a_1 d(Iu, Jx_{2n+1}) + a_2 d(Iu, Su) + a_3 d(Jx_{2n+1}, Tx_{2n+1}) + a_4 d(Iu, Tx_{2n+1}) + a_5 d(Jx_{2n+1}, Su) + d(y_{2n+1}, Iu) \\ &= a_1 d(Iu, y_{2n}) + a_2 d(Iu, Su) + a_3 d(y_{2n}, y_{2n+1}) + a_4 d(Iu, y_{2n+1}) + a_5 d(y_{2n}, Su) + d(y_{2n+1}, Iu) \\ &= a_1 d(Iu, y_{2n}) + a_2 d(Iu, Su) + a_3 [d(y_{2n}, Iu) + d(Iu, y_{2n+1})] + a_4 d(Iu, y_{2n+1}) + a_5 [d(y_{2n}, Iu) + d(Iu, Su)] \\ &\quad + d(y_{2n+1}, Iu) \end{aligned}$$

$$d(Su, Iu)(1 - a_2 - a_5) \leq (a_1 + a_3 + a_5) d(Iu, y_{2n}) + (a_3 + a_4 + 1) d(Iu, y_{2n+1})$$

As $y_{2n} \rightarrow Iu$, $y_{2n+1} \rightarrow Iu$ and $(1 - a_2 - a_5) > 0$, using Lemma 2.2, we have $d(Su, Iu) = 0$, and we get $Iu = Su$. Thus $Iu = Su = z$. Therefore z is a point of coincidence of the pair (S, I) . Since (S, I) is weakly compatible, $Sz = Iz$.

As $S(X) \subseteq J(X)$, there exists $w \in X$ such that $z = Sz = Jw$. So $z = Su = Iu = Jw$.
 (1.5)

Put $x = u$ and $y = w$ in (3.1.5) we get,

$$d(Su, Tw) \leq a_1 d(Iu, Jw) + a_2 d(Iu, Su) + a_3 d(Jw, Tw) + a_4 d(Iu, Tw) + a_5 d(Jw, Su)$$

by using (1.5) we have,

$$d(z, Tw) \leq a_1d(z, z) + a_2d(z, z) + a_3d(z, Tw) + a_4d(z, Tw) + a_5d(z, z)$$

$$d(z, Tw) \leq (a_3 + a_4) d(z, Tw).$$

As $(a_3 + a_4) < 1$, using Proposition 2.1, it follows that $d(z, Tw) = 0$ and we get $Tw = z$. As the pair (T, J) is weakly compatible we get $Tz = Jz$.

Put $x = z$ and $y = z$ in (3.1.5) and using $Tz = Jz, Sz = Iz$

$$d(Sz, Tz) \leq a_1d(Iz, Jz) + a_2d(Iz, Sz) + a_3d(Jz, Tz) + a_4d(Iz, Tz) + a_5d(Jz, Sz)$$

$$d(Sz, Tz) \leq a_1d(Sz, Tz) + a_2d(Iz, Iz) + a_3d(Jz, Jz) + a_4d(Sz, Tz) + a_5d(Tz, Sz)$$

$$d(Sz, Tz) \leq (a_1 + a_4 + a_5) d(Sz, Tz).$$

As $(a_1 + a_4 + a_5) < 1$ we get, $Sz = Tz$, by proposition 1 and we have $Sz = Tz = Iz = Jz$.

Thus z is a point of coincidence of the four self maps S, T, I and J .

Case II. $J(X)$ is complete. The proof of this case is similar to Case I.

Case III. $S(X)$ is complete. Since $\{y_n\}$ is a Cauchy sequence in X . It follows that $(y_{2n} = Sx_{2n})$ is a Cauchy sequence in $S(X)$, which is complete. So $y_{2n} \rightarrow z = Sv$ for some $v \in X$. Now $S(X) \subseteq J(X)$, there exists $p \in X$ such that $z = Sv = Jp$. It follows from Case II that $Sz = Tz = Iz = Jz$. Thus, also in this case, the maps S, T, I and J have a common fixed point of coincidence.

Case IV. $T(X)$ is complete. The proof of this case is similar to case III.

We have $z = Tz = Iz$. Let $Su = Iu$ be another point of coincidence of the pair (S, I) .

$$d(Su, z) \leq d(Tx_{2n+1}, z) + d(Tx_{2n+1}, Su)$$

$$= d(y_{2n+1}, z) + d(Su, Tx_{2n+1})$$

Put $x = u$ and $y = x_{2n+1}$ in (e) we get

$$d(Su, z) \leq d(y_{2n+1}, z) + a_1d(Iu, Jx_{2n+1}) + a_2d(Iu, Su) + a_3d(Jx_{2n+1}, Tx_{2n+1}) + a_4d(Iu, Tx_{2n+1}) +$$

$$a_5d(Jx_{2n+1}, Su)$$

$$= d(y_{2n+1}, z) + a_1d(Iu, y_{2n}) + a_3d(y_{2n}, y_{2n+1}) + a_4d(Iu, y_{2n+1}) + a_5d(y_{2n}, Su)$$

$$= d(y_{2n+1}, z) + a_1 [d(Su, z) + d(z, y_{2n})] + a_3 [d(y_{2n}, z) + d(z, y_{2n+1})] + a_4 [d(Su, z) + d(z, y_{2n+1})] +$$

$$a_5 [d(y_{2n}, z) + d(z, Su)]$$

$$(1 - a_1 - a_4 - a_5)d(Su, z) \leq (1 + a_3 + a_4) d(z, y_{2n+1}) + (a_1 + a_3 + a_5)d(z, y_{2n})$$

Since $y_{2n+1} \rightarrow z$ and $y_{2n} \rightarrow z$, and we have $(1 - a_1 - a_4 - a_5) > 0$, using Lemma 2 we obtain $d(z, Su) = 0$ and so $Su = z$. Hence the point of coincidence of (S, I) is unique. As the pair (S, I) is weakly compatible by Lemma 1 z is the unique common fixed point of S and I .

Hence $z = Sz = Iz = Jz = Tz$ is the unique fixed point of S, T, I and J .

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