



# Computational algorithm for fractional Fredholm integro-differential equations

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## Abstract

Fractional calculus is a fascinating field of mathematics that focuses on the study of integrals and derivatives of arbitrary orders, extending the principles of basic calculus. Its applications span across various scientific, engineering, and other disciplines. In this study, the collocation method, in conjunction with the utilization of the fourth kind Chebyshev polynomials, is employed to explore solutions for fractional integro-differential equations of Fredholm type. By applying the collocation method, the problem at hand is transformed into a system of linear algebraic equations. These equations are subsequently solved by employing matrix inversion techniques to determine the unknown constants. To provide a comprehensive understanding and visualization of the results, the research incorporates tables and figures, which present numerical examples and comparisons. These comparisons serve to highlight the superior performance of the proposed method in terms of efficiency and convenience when compared to traditional methods. By showcasing the advantages of the collocation method and the utilization of fourth kind Chebyshev polynomials, the research underscores the potential of these approaches in solving fractional integro-differential equations.

**AMS subject classifications:** Primary 45D05; Secondary 42C10, 65G99

**Keywords:** Caputo derivative; Fredholm fractional integro-differential equations; Collocation technique; Approximate solution

## 1. Introduction

The field of fractional calculus emerges as a natural extension of the conventional calculus integral and derivative operators, analogous to how fractional exponents stem from integer exponents. Unlike regular calculus, which restricts derivatives to natural numbers, fractional calculus allows for derivatives of any real order. This versatility makes fractional calculus highly applicable across a wide range of scientific and engineering domains. Fractional calculus proves invaluable in modeling various scientific phenomena, including image processing, earthquake engineering, biomedical engineering, computational fluid dynamics, heat distribution in furnaces, virus propagation, and satellite positioning in space. Notably, references [1,2,3,4] provide comprehensive insights into the fundamental concepts of fractional calculus and its applications across multiple disciplines. Due to the inherent complexity of Fractional Integro-Differential Equations (FIDEs), analytical solutions are often elusive, necessitating the exploration of approximation and numerical methods. Previous studies have utilized different approaches, such as Legendre-Gauss quadrature combined with Lucas wavelets [5], Laguerre polynomials [6,7], semi-analytical methods [8], Bernstein modified homotopy perturbation approach [9], collocation techniques with various basis functions [10,11,12], approximate solutions using Volterra-Fredholm IDEs [13], Sumudu transform method and Hermite spectral collocation method [14], and investigation of fractional derivatives in nonlinear reaction-

diffusion equations [15]. The numerical solution of fractional singular IDEs has been explored using the Galerkin method, Taylor series expansion, and Chebyshev polynomials [16, 17]. Researchers have investigated the numerical solution of the fractional Benney equation [18], employed the least-squares method [19, 20], and introduced the sinc-collocation approach for linear Fredholm IDEs [21]. Linear fractional Fredholm IDEs were addressed using second-kind Chebyshev wavelets [22], while numerical techniques were applied to solve nonlinear integro-differential equations [23]. The Sinc-Galerkin method was utilized to tackle space-fractional boundary value problems [24]. Additionally, the numerical solution of fractional integro-differential problems was achieved using cubic B-spline wavelets [25]. The homotopy analysis transform approach was employed for efficient solutions of FIDEs [26], and a comparative examination of numerical techniques for fractional IDEs was conducted [27]. Inspired by the aforementioned research, this study introduces a computational algorithm that employs fourth-kind Chebyshev polynomials as basis functions for solving FIDEs. This method offers improved accuracy while reducing computational costs, thus presenting a promising approach to efficiently address FIDEs.

$$\zeta''(z) + \tau(z)\zeta'(z) + \eta(z) {}_a D^\alpha \zeta(z) + r(z)\zeta(z) = f(z) + \lambda \int_a^b K(z, t)\zeta(t)dt, \quad (1)$$

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$$\eta(z)_{\alpha} \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-t)^{n-\alpha-1} \frac{d^n}{dt^n} \left( \sum_{i=0}^n \omega^*(t) a_i \right) dt +$$

$$r(z) \sum_{i=0}^n \omega^*(z) a_i = f(z) + \lambda \int_a^b K(z,t) \sum_{i=0}^n \omega^*(t) a_i dt, \quad (6)$$

$$\text{Let } \iota(z) = \sum_{i=0}^n \omega^{*''}(z) a_i, \quad \varpi(z)$$

$$= \tau(z) \sum_{i=0}^n \omega^{*'}(z) a_i, \quad \sigma(z) = r(z) \sum_{i=0}^n \omega^*(z) a_i$$

$$\epsilon(z) = \eta(z)_{\alpha} \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-t)^{n-\alpha-1} \frac{d^n}{dt^n} \left( \sum_{i=0}^n \omega^*(t) a_i \right) dt,$$

$$q(z) = \lambda \int_a^b K(z,t) \sum_{i=0}^n \omega^*(t) a_i dt,$$

Substituting  $\iota(z), \varpi(z), \sigma(z), \zeta(z)$  and  $\eta(z)$  in equation (6), gives

$$\iota(z) + \varpi(z) + \sigma(z) + \epsilon(z) = f(z) + q(z) \quad (7)$$

Collocating (7) at equally spaced point  $z_i = a + \frac{(b-a)i}{n}, i = 0, 1, 2, \dots, n$  gives linear system algebraic of equations in  $(n+1)$  unknown constants  $a_i$ 's. Additional equations are obtained from equation (2), which are represented in matrix form:

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & \cdots & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & \cdots & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1} & P_{m2} & P_{m3} & \cdots & \cdots & \cdots & P_{mn} \\ P_{11}^0 & P_{12}^0 & P_{13}^0 & \cdots & \cdots & \cdots & P_{1n}^0 \\ P_{21}^1 & P_{22}^1 & P_{23}^1 & \cdots & \cdots & \cdots & P_{2n}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1}^{n-1} & P_{m2}^{n-1} & P_{m3}^{n-1} & \cdots & \cdots & \cdots & P_{mn}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Q_{11} \\ Q_{22} \\ \vdots \\ \vdots \\ \vdots \\ Q_{mn} \\ Q_{11}^0 \\ Q_{22}^1 \\ \vdots \\ \vdots \\ \vdots \\ Q_{mn}^{n-1} \end{pmatrix} \quad (8)$$

where  $P_i'$ s and  $P_i^{n-1}$ 's are the coefficients of  $a_i$ 's and  $Q_i'$ s are values of  $f(z_i)$ .

The matrix inversion approach is then used to solve the system of equations in order to obtain the unknown constants.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & \cdots & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & \cdots & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1} & P_{m2} & P_{m3} & \cdots & \cdots & \cdots & P_{mn} \\ P_{11}^0 & P_{12}^0 & P_{13}^0 & \cdots & \cdots & \cdots & P_{1n}^0 \\ P_{21}^1 & P_{22}^1 & P_{23}^1 & \cdots & \cdots & \cdots & P_{2n}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1}^{n-1} & P_{m2}^{n-1} & P_{m3}^{n-1} & \cdots & \cdots & \cdots & P_{mn}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} Q_{11} \\ Q_{22} \\ \vdots \\ \vdots \\ \vdots \\ Q_{mn} \\ Q_{11}^0 \\ Q_{22}^1 \\ \vdots \\ \vdots \\ \vdots \\ Q_{mn}^{n-1} \end{pmatrix} \quad (9)$$

Solving equation (9) yields the unknown constants, which are then substituted into the assumed approximate solution in equation (5) to obtain the required approximate solution.

#### 4. Numerical examples

**Example 1.** Consider the fractional Fredholm Integro-differential shown below [33].

$$\zeta''(z) + D^{\alpha} \zeta(z) + \zeta(z) = f(z) + 2 \int_0^1 K(z,t) \zeta(t) dt, \quad (10)$$

Subject to  $\zeta(0) = 0, \zeta(1) = 0$ , for  $\alpha = \frac{1}{2}$ ,

where

$$f(z) = -6z + z(1-z^2) + \frac{2z^{0.5} - 3.3z^{2.5}}{\Gamma(0.5)}$$

$$K(z,t) = \frac{e}{(1+z^{\frac{1}{2}})^2} e^{-t}$$

The exact solution is  $\omega(z) = z - z^3$ .

Applying the proposed technique for different values of  $\alpha = 0.5, 0.6, 0.7, 0.8$ , we have the following approximate solutions.

For  $\alpha = 0.5$ ,

$$\begin{aligned} \zeta(z) = & 1.000000001s - 0.0000000005539314020z^2 \\ & - 6.773549718 \times 10^{-11} - 0.000007969168026z^5 \\ & + 0.000002216216189z^6 - 0.000003543763114z^7 \\ & - 1.000000013z^3 + 0.000003272789627z^8 \\ & - 0.000001623419848z^9 + 0.0000001525225959z^4 \\ & + 0.0000003349596227z^{10} \end{aligned}$$

For  $\alpha = 0.6$ ,

$$\begin{aligned} \zeta(z) = & 1.012516177z + 1.878206717z^4 \\ & + 8.475917199z^8 - 1.399509651z^3 \\ & - 11.48751572z^7 - 0.000736567728z^2 \\ & - 5.244739592z^5 + 9.661026652z^6 \\ & - 3.528066395z^9 + 0.3684479231 \times 10^{-5} \\ & + 0.6328959819z^{10} \end{aligned}$$

For  $\alpha = 0.7$ ,

$$\begin{aligned} \zeta(z) = & 1.022608544z + 4.473556998z^4 \\ & + 0.00001370788345 + 20.72876377z^8 \\ & - 1.914103134z^3 - 28.03704528z^7 \\ & - 0.00218649120z^2 - 12.68998686z^5 \\ & + 23.50734227z^6 - 8.640655837z^9 \\ & + 1.551685353z^{10} \end{aligned}$$

For  $\alpha = 0.8$ ,

$$\begin{aligned} \zeta(z) = & 1.028909687z + 7.925116395z^4 \\ & + 37.78530153z^8 - 2.555535709z^3 \\ & - 50.99064098z^7 - 0.00446864518z^2 \\ & - 22.85418843z^5 + 42.60511321z^6 \\ & - 15.77619846z^9 + 2.836532123z^{10} \\ & + 0.00003768357627 \end{aligned}$$

For  $\alpha = 0.9$ ,

$$\begin{aligned} \zeta(z) = & 1.029802479z + 12.31769516z^4 \\ & + 60.56758193z^8 - 3.322656232z^3 \\ & - 81.52267696z^7 - 0.00767260138z^2 \\ & - 36.13681627z^5 + 67.84838188z^6 \\ & - 25.33547416z^9 + 4.561689378z^{10} \\ & + 0.00008982173221 \end{aligned}$$

**Example 2.** Consider the fractional Fredholm Integro-differential shown below [33].

$$\zeta''(z) - z^2 D^\alpha \zeta(z) + z\zeta(z) = f(z) - \int_0^1 K(z,t)\zeta(t)dt, \quad (11)$$

Subject to  $\zeta(0) = 0, \zeta(1) = 0$ , for  $\alpha = 0.3$ , where

$$\begin{aligned} f(z) = & -z^5 + s^3 - 12z^2 + \frac{4z}{15} + \frac{68}{35} \\ & + \frac{24}{\Gamma(4.7)}z^{5.7} - \frac{2}{\Gamma(2.7)}z^{3.7} \\ K(z,t) = & 2z - z^2 \end{aligned}$$

The exact solution is  $\zeta(z) = z^2(1 - z^2)$ .

Applying the proposed technique for different values  $\alpha = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ , we have the following approximate solutions.

For  $\alpha = 0.3$ ,

$$\begin{aligned} \zeta(z) = & -0.000000001113458825z + 0.9999999997z^2 \\ & + 0.00000004940973504z^3 + 0.000008655725195z^7 \\ & - 1.000000494z^4 - 0.000007565990700z^8 \\ & - 1.412625578 \times 10^{-11} - 0.000005785864943z^6 \\ & + 0.000003577554707z^9 + 0.000002271521353z^5 \\ & - 0.0000007071503205z^{10} \end{aligned}$$

For  $\alpha = 0.4$ ,

$$\begin{aligned} \zeta(z) = & -0.0003980866494z + 1.000055421z^2 \\ & - 0.00002298244777z^3 + 0.005338570547z^6 \\ & - 0.01244940592z^7 + 0.005627248008z^5 \\ & - 1.000350638z^4 + 0.002203273777z^8 \\ & - 0.0002416560346z^{10} + 0.0002382558454z^9 \\ & - 5.795053600 \times 10^{-12} \end{aligned}$$

For  $\alpha = 0.5$ ,

$$\begin{aligned} \zeta(z) = & -0.0003157163139z + 1.000157321z^2 \\ & - 0.0001172215148z^3 + 0.008273388952z^6 \\ & - 0.02558403537z^7 + 0.01340702138z^5 \\ & - 1.000779210z^4 + 0.005203663772z^8 \\ & - 0.0005608165666z^{10} + 0.000315605515z^9 \\ & - 1.552820800 \times 10^{-12} \end{aligned}$$

For  $\alpha = 0.6$ ,

$$\begin{aligned} \zeta(z) = & 0.0004128464578z + 1.000318755z^2 \\ & - 0.0003072690378z^3 + 0.007334296146z^6 \\ & - 0.03806329378z^7 + 0.02380888838z^5 \\ & - 1.001237780z^4 + 0.008297688875z^8 \\ & - 0.001023542478z^{10} + 0.000459410337z^9 \\ & - 7.251342900 \times 10^{-12} \end{aligned}$$

For  $\alpha = 0.7$ ,

$$\begin{aligned} \zeta(z) = & 0.001998437927z + 1.000556090z^2 \\ & - 0.0006241632762z^3 + 0.00069385209z^6 \\ & - 0.04785095081z^7 + 0.03728808048z^5 \\ & - 1.001636049z^4 + 0.01014946485z^8 \\ & - 0.001765448447z^{10} + 0.001190686564z^9 \\ & - 6.834071000 \times 10^{-12} \end{aligned}$$

For  $\alpha = 0.8$ ,

$$\begin{aligned} \zeta(z) = & 0.004712134595z + 1.000890101z^2 \\ & - 0.001106362635z^3 - 0.01384708196z^6 \\ & - 0.05202184980z^7 + 0.05423713877z^5 \\ & - 1.001829107z^4 + 0.00849687675z^8 \\ & - 0.003043230907z^{10} + 0.00351138094z^9 \\ & - 2.833750000 \times 10^{-13} \end{aligned}$$

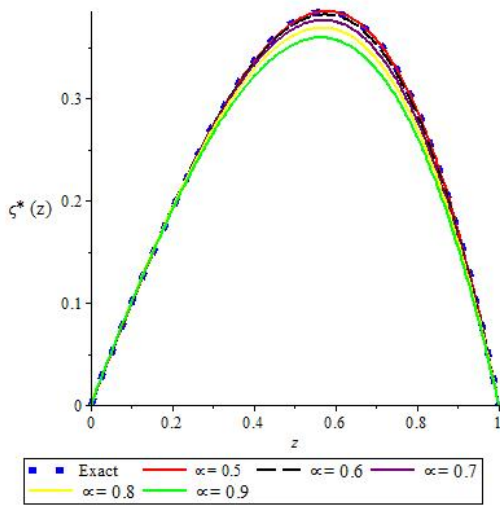


Figure 1: Approximation solutions of example 3.

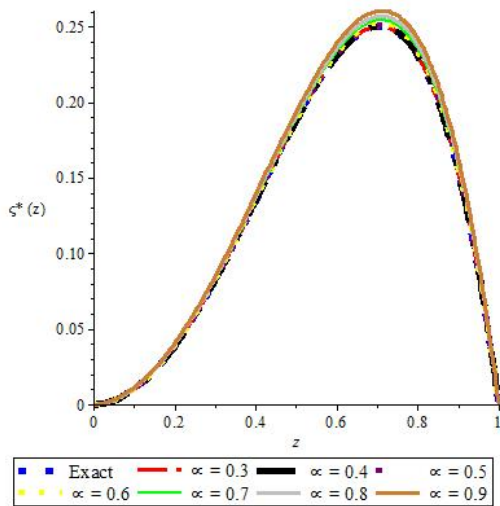


Figure 2: Approximation solutions of example 2.

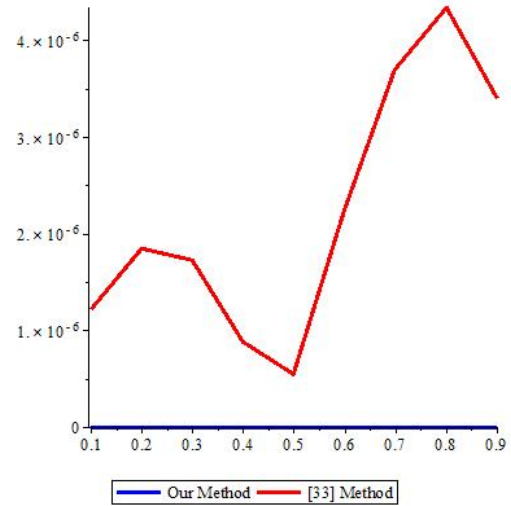


Figure 3: Comparison of the absolute errors of Example 1.

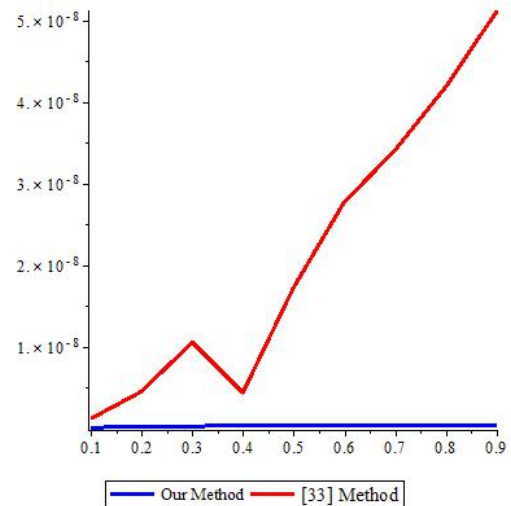


Figure 4: Comparison of the absolute errors of Example 2.

For  $\alpha = 0.9$ ,

$$\begin{aligned} \zeta(z) = & 0.008907426096z + 1.001347608z^2 \\ & - 0.001800170595z^3 - 0.03889766815z^6 \\ & - 0.04647314646z^7 + 0.07494111780z^5 \\ & - 1.001610728z^4 - 0.00030247757z^8 \\ & - 0.005323318470z^{10} + 0.00921135674z^9 \\ & - 2.018833000 \times 10^{-12} \end{aligned}$$

## 5. Numerical results

The numerical results of Example 1 and Example 2 are shown in Table 1 and Table 2 respectively.

## 6. Discussion of results

In the comparative analysis, we examined the results obtained from the sinc interpolation and quadratic approaches presented in reference [33]. Upon comparing these findings with our proposed method, it becomes evident, as illustrated in Table 1 and 2, that our method significantly outperforms the results reported in [33]. To provide further evidence of the accuracy achieved by our

proposed method, Fig. 1 showcases the exceptional agreement between the approximate solutions and the exact solutions at  $\alpha$  values of  $\alpha = 0.5$  and  $0.6$ . Notably, as the value of alpha increases, ranging from  $\alpha = 0.7$  to  $0.9$ , the curve flattens out. Fig. 2 offers additional insights, demonstrating the close correspondence between the approximate solutions and the exact solution for  $\alpha$  values of  $0.3, 0.4,$  and  $0.5$ . However, as the  $\alpha$  value further increases, specifically from  $0.7$  to  $0.9$ , the curve becomes steeper. Furthermore, by comparing the absolute errors between our proposed method and the errors reported in [33], as depicted in Fig. 3 and 4, it becomes evident that our approach yields significantly smaller errors. This indicates the superior accuracy and reliability of our method in approximating the exact solutions. Fig. 1 and 2 not only highlight the accuracy of our proposed method but also emphasize the potential for utilizing different  $\alpha$  values as a means of controlling systems. The variations in the  $\alpha$  parameter allow for fine-tuning and adjusting the behavior of the system under consideration.

## 7. Conclusion

In this study, we employed a robust collocation computational algorithm to obtain numerical solutions for fractional integro-differential equations. By implementing this method, we achieved significantly higher accuracy compared to the results obtained in

**Table 1:** Numerical results for Example 1.

s	Exact	Our Method AS	[33] AS	Our Method AE	[33] AE
0.0	0	0	0	0	0
0.1	0.099	0.09900000000000	0.098998	0	1.21E - 06
0.2	0.192	0.19200000000000	0.191998	0	1.85E - 06
0.3	0.273	0.27300000020000	0.272998	1.557E - 10	1.73E - 06
0.4	0.3360	0.33600000010000	0.335999	1.817E - 10	8.77E - 07
0.5	0.375	0.37500000010000	0.375001	1.812E - 10	5.51E - 07
0.6	0.384	0.38400000010000	0.384002	1.560E - 10	2.23E - 06
0.7	0.357	0.35700000010000	0.357004	9.694E - 11	3.70E - 06
0.8	0.288	0.28800000000000	0.288004	0	4.34E - 06
0.9	0.171	0.17099999990000	0.171003	1.018E - 10	3.40E - 06
1.0	0	0	0	0	0

AS: Approximate Solution, AE: Absolute Error

**Table 2:** Numerical results for Example 2.

s	Exact Solution	Our Method AS	[33] AS	Our Method AE	[33] AE
0.0	0	0	0	0	0
0.1	0.0099	0.00989999989000	0.009899	1.107E - 10	1.15E - 09
0.2	0.0384	0.03839999982000	0.038399	1.942E - 10	4.51E - 09
0.3	0.0819	0.08189999972000	0.081899	2.778E - 10	1.05E - 08
0.4	0.1344	0.13439999980000	0.134399	3.434E - 10	4.40E - 09
0.5	.1875	0.18749999950000	0.187499	3.987E - 10	1.72E - 08
0.6	0.2304	0.23039999960000	0.230399	4.490E - 10	2.76E - 08
0.7	0.2499	0.24989999950000	0.2498999	4.652E - 10	3.40E - 08
0.8	0.2304	0.23039999960000	0.230399	4.231E - 10	4.20E - 08
0.9	0.1539	0.15389999970000	0.153899	3.426E - 10	5.12E - 08
1.0	0	0	0	0	0

AS: Approximate Solution, AE: Absolute Error

the two cases presented in [33]. Our findings were rigorously validated through extensive numerical calculations, which demonstrated excellent agreement between our numerical solutions and the exact solutions. Based on the remarkable accuracy and reliability demonstrated by our proposed collocation computational algorithm, we highly recommend its adoption for addressing other fractional integro-differential equations of higher order. The method's effectiveness in yielding precise results positions it as a valuable tool for tackling complex problems in the realm of fractional calculus. Researchers and practitioners alike can benefit from leveraging this technique to obtain accurate solutions for a wide range of fractional integro-differential equations.

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