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OPTIMALITY CONDITION AND DUALITY IN MULTI OBJECTIVE PROGRAMMING WITH GENERALIZED (φ, ρ)-UNIVEXITY.

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ABSTRACT

In this paper, we extend the classes of generalized type I vector valued functions introduced by Aghezzaf and Hachimi[1] to generalized univex type I vector-valued functions and consider a multiobjective optimization problem involving generalized type I function with (φ, ρ) -univexity. A number of Kuhn-Tucker type sufficient optimality conditions are obtained for a feasible solution to be an efficient solution. The Mond-Weir and general Mond-Weir type duality results are also presented.

1. INTRODUCTION

Rueda *et al.*[2] obtained optimality and duality results for several mathematical programs by combining the concept of type I functions and univex functions [3]. Mishra[4] obtained optimality. duality and saddle point results for a multiple–objective program by combining the concept of pseudoquasi, type I , quasi-pseudo type I, strictly pseudoquasi, type I and univex functions. Mishra *et al.*[5] introduce new class of generalized type I univex functions by extending weak strictly pseudoquasi type I, strong pseudoquasi type I etc.Recently Caristi, Ferrara and Stefanescu[6] introduced (φ, ρ) invexity.

In this paper, we introduce new class of generalized type I univex functions with (φ, ρ) univexity and also studied weak strictly pseudoquasi type I,strong pseudoquasi type I, weak quasistrictly–pseudo type I and weak strictly pseudo type I. In section 2,we introduce some preliminaries. Some sufficient optimality results are established in section 3. A number of duality theorems in the Mond-Weir type are shown in section 4.In section 5. We are giving two results on general Mond-Weir type duality.

2. PRELIMINARIES

To compare vectors along the lines of Mangasarian [7],we will distinguish between $\leq$ *and* $\leqq$ *or between* $\geq$ *and* $\geqq$ specifically. $x \in R^n, y \in R^n, x \leq y \Leftrightarrow x_i \leqq y_i \forall i = 1, 2, ...n, x \neq y,$

Similarly notations are applied to distinguish between $\geq$ *and* $\geqq$.

We consider the following multiple objective optimization problem:

(VP) minimize $f(x) = (f_1(x)..........f_p(x))$ *subject to* $g(x) \leqq x, x \in X \subseteq R^n.$

where $f : X \to R^p$ *and* $g : X \to R^m$ are differentiable functions and $X \subseteq R^n$ is an open set.

Let X_0 be the set of all feasible solutions of (VP). We quote some definitions and also give some new ones.

Definition 2.1

A point $a \in X_0$ is said to be an efficient solution of problem (VP) if there exit no $x \in X_0$ such that $f(x) \leq f(a), f(x) \neq f(a).$

Definition 2.2

A point $a \in X_0$ is said to be a weakly efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition 2.3

A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant K such that for each $x \in X_0$ and for each $i \in \{1, 2......p\}$ *satisfying* $f_i(x) < f_i(a)$, there exist at least one $i \in \{1, 2......p\}$ *such that* $f_j(a) \; _j \; f$ and $f_i(a) - f_i(x) \leq K\left(f_j(x) - f_j(a)\right)$.

Denoting by WE(VP), E(VP) and PE(VP) the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have PE(VP) $\subseteq$ E(VP) $\subseteq$ WE(VP).

For convenience, let us write the definitions of (Φ, ρ)-univexity on the lines from[1], Let $\varphi : X_0 \to R$ be a differentiable function $(X_0 \subseteq R^n)$, $X \subseteq X_0$, and $a \in X_0$. An element of all (n+1)- dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ *and* $r \in R, \rho$ is a real number and Φ is real valued function defined on $X_0 \times X_0 \times R^{n+1}$, *such that* $\varphi(x, a, .)$ *is* convex on R^{n+1} *and* $\Phi(x, a, (0, r)) \geq 0$, for every $(x, a) \in X_0 \times X_0$ and $r \in R_+$. $b_0, b_1 : X \times X \times [0,1] \to R_+$ $b(x,a) = \lim_{\lambda \to 0} b(x, a, \lambda) \geq 0$, and b does not depend upon λ if the corresponding functions are differentiable. $\psi_0, \psi_1 : R \to R$ is an n-dimensional vector- valued function.

We assume that $\psi_0, \psi_1 : R \to R$ satisfying $u \leq 0 \Rightarrow \psi_0(u) \leq 0$ *and* $u \leqq 0 \Rightarrow \psi_1(u) \leqq 0$, *and* $b_0(x,a) > 0$ *and* $b_1(x,a) \geqq 0$. and $\psi_0(\alpha) = -\psi_0(\alpha)$ and $\psi_1(-\alpha) = -\psi_1(\alpha)$.

Example 2.1[6]

min $f(x) = x - 1$

$g(x) = -x - 1 \leq 0, x \in X_0 \in [1, \infty)$

$\Phi(x, a; (y, r)) = 2(2^r - 1)|x - a| + \langle y, x - a \rangle$ for $\psi_0(x) = x$, $\psi_1(x) = -x$, $\rho_1 = \frac{1}{2}$ (for f) and $\rho = 1$ (for g), then this is (ϕ, ρ)-univex but it is not (ϕ, ρ)-invex .

Definition 2.4

The problem(VP) is said to be weak strictly pseudo type I univex at $a \in X_0$ if there exit real – valued functions $b_0, b_1 . \psi_0, \psi_1$ *and* ρ *such that*

$b_0(x,a)\psi_0[f(x) - f(a)] \leq 0 \Rightarrow \varphi(x, a, (\nabla f(a), \rho)) < 0.$

$-b_1(x,a)\psi_1[g(a)] \leqq 0 \Rightarrow \varphi(x, a, (\nabla g(a), \rho)) \leqq 0.$

for all $x \in X_0$ and for all $i = 1, 2......p$ *and* $j = 1, 2.......m$. If (VP) is weakly strictly pseudo type I (ϕ, ρ)- univex at each $a \in X$, (VP) is said to be weak strictly pseudo type I (ϕ, ρ)-univex on X.

Remark 2.1[5]
There exist functions which are weak strictly pseudoquasi type I univex, with respect to $b_0 = 1 = b_1$, ψ_0 and ψ_1 are identity function on R, but not strictly pseudoquasi type I univex, with respect to same $b_0, b_1, \psi_0, \psi_1, \rho$.

Definition 2.5.
The problem (VP) is said to be strong pseudoquasi type I (φ, ρ)- univex at $a \in X_0$ at if there exit real- valued functions $b_0, b_1.\psi_0, \psi_1$ *and* ρ *such that*

$$b_0(x,a)\psi_0[f(x) - f(a)] \leq 0 \Rightarrow \phi(x,a,(\nabla f(a),\rho)) \leq 0.$$

$$-b_1(x,a)\psi_1[g(a)] \leqq 0 \Rightarrow \phi(x,a,(\nabla g(a),\rho)) \leqq 0.$$

for all $x \in X_0$ and for all $i = \{1,2......P\}$ *and* $j = \{1,2.......m\}$. if (VP) is strong pseudoquasi type I(φ, ρ) univex at each $a \in X$, (VP) is said to strong pseudoquasi type I (φ, ρ)-univex on X.

Remark 2.2[5]
There exist functions which are strong pseudoquasi type I univex with respect to $b_0 = 1 = b_1$, ψ_0 and ψ_1 are identity function on R, but not weak strictly pseudoquasi type I univex with respect to same $b_0, b_1, \psi_0, \psi_1, \rho$.

Definition 2.6
The problem (VP) is weak quasi strictly Pseudo type I (φ, ρ)- univex with respect to $b_0, b_1.\psi_0, \psi_1$ *and* ρ *at* $a \in X_0$ if there exit real-valued functions $b_0, b_1.\psi_0, \psi_1$ *and* ρ such that

$$b_0(x,a)\psi_0[f(x) - f(a)] \leq 0 \Rightarrow \phi(x,a,(\nabla f(a),\rho)) < 0.$$

$$-b_1(x,a)\psi_1[g(a)] \leqq 0 \Rightarrow \phi(x,a,(\nabla g(a),\rho)) < 0.$$ for all $x \in X_0$ and for all $i = \{1,2......p\}$ *and* $j = \{1,2.......m\}$. If (VP) is weak quasi strictly pseudo type I univex at each $a \in X$, (VP) is said to be weak quasi strictly pseudo type I (φ, ρ)- univex on X.

Definition 2.7
Weak strictly pseudo type I (φ, ρ)- univex with respect to $b_0, b_1.\psi_0, \psi_1$ *and* ρ *at* $a \in X_0$ if there exit real-valued functions $b_0, b_1.\psi_0, \psi_1$ *and* ρ such that

$$b_0(x,a)\psi_0[f(x) - f(a)] \leq 0 \Rightarrow \phi(x,a,(\nabla f(a),\rho)) < 0.$$

$$-b_1(x,a)\psi_1[g(a)] \leqq 0 \Rightarrow \phi(x,a,(\nabla g(a),\rho)) < 0.$$

for all $x \in X_0$ and for all $i = \{1,2......p\}$ *and* $j = \{1,2.......m\}$. If (VP) is weak strictly pseudo type I univex at each $a \in X$, (VP) is said to be weak strictly pseudo type I (φ, ρ)- univex on X.

3. OPTIMALITY CONDITIONS

In this section, we establish some sufficient optimality condition for an $a \in X_0$ to be an efficient solution of problem (VP) under various generalized type I (φ, ρ)- univex functions defined in the previous section.

Theorem 3.1 (sufficiency) Suppose that
(i) $a \in X_0$ (ii) There exist $\tau^0 \in R^p, \tau^0 > 0, \lambda \in R^m$ *and* $\lambda^0 \geqq 0$ Such that
(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$ (b) $\lambda^0 g(a) = 0$ (c) $\tau^0 e = 1$, *where* $e = (1,......1)^T \in R^P$;

(iii)The problem (VP) is strong pseudoquasi type I (φ,ρ)- univex at $a \in X_0$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ for all feasible x. then a is an efficient solution to (VP).

Proof

Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that $f(x) \le f(a)$.

By the properties of b_0 *and* ψ_0 and the above inequality, we have $b_0(x,a)\psi_0[f(x)-f(a)] \le 0$ (1)

By the feasibility of a, we have $-\lambda^0 g(a) \leqq 0$

By the properties of b_1 *and* ψ_1 and the above inequality,

we have $-b_1(x,a)\psi_1[\lambda^0 g(a)] \leqq 0$ (2)

By inequalities (1) and (2) and condition (iii), we have

$\phi(x,a;(\nabla f(a),\rho)) \le 0$ *and* $\phi(x,a;(\lambda^0 \nabla g(a),\rho)) \leqq 0$, *Since* $\tau^0 > 0$, the above inequalities give

$\phi(x,a;(\tau^0 \nabla f(a) + \lambda^0 \nabla g(a),\rho)) < 0$ (3)

which contradict condition (iii). This completes the proof.

Theorem 3.2 (sufficiency) Suppose that

(i) $a \in X_0$ (ii) There exist $\tau^0 \in R^p, \tau^0 \ge 0, \lambda \in R^m$ *and* $\lambda^0 \geqq 0$ Such that

(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$ (b) $\lambda^0 g(a) = 0$ (c) $\tau^0 e = 1$, *where* $e = (1,......1)^T \in R^P$;

(iii) The problem (VP) is weak strictly pseudoquasi type I (φ,ρ)- univex at $a \in X_0$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ for all feasible x. then a is an efficient solution to (VP).

Proof

Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that $f(x) \le f(a)$.

By the property of b_0 *and* ψ_0 and the above inequality, we get (1). By the feasibility of a the properties of b_1 *and* ψ_1 and the condition (iii), we have

$\phi(x,a;(\nabla f(a),\rho)) < 0$ *and* $\phi(x,a;(\lambda^0 \nabla g(a),\rho)) \leqq 0$, *Since* $\tau^0 \ge 0$, the above inequalities give

$\phi(x,a;(\tau^0 \nabla f(a) + \lambda^0 \nabla g(a),\rho)) < 0$

which contradicts (iii). This completes the proof.

Theorem 3.3 (sufficiency) Suppose that

(i) $a \in X_0$ (ii) There exist $\tau^0 \in R^p, \tau^0 \geqq 0, \lambda \in R^m$ *and* $\lambda^0 \geqq 0$ Such that

(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$ (b) $\lambda^0 g(a) = 0$ (c) $\tau^0 e = 1$, *where* $e = (1,......1)^T \in R^P$;

(iii) The problem (VP) is weak strictly pseudo type I (φ,ρ)- univex at $a \in X_0$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ for all feasible x, then a is an efficient solution to (VP).

Proof

Suppose contrary to the result that a is not an efficient solution to (VP). Then there exists a feasible solution x to (VP) such that $f(x) \le f(a)$.

By the property of b_0 *and* ψ_0 and the above inequality, we get (1). By the feasibility of a and properties of b_1 *and* ψ_1 we get (2). By inequalities (1) and (2) and condition (iii) , we have

$\phi(x,a;(\nabla f(a),\rho)) < 0$ *and* $\phi(x,a;(\lambda^0 \nabla g(a),\rho)) < 0$, *Since* $\tau^0 \geqq 0$, the above inequalities give

$\phi(x,a;(\tau^0 \nabla f(a) + \lambda^0 \nabla g(a),\rho)) < 0$ which contradicts (iii). This completes the proof.

4. MOND-WEIR TYPE DUALITY

In this section, we present some weak and strong duality theorems for (VP) and the following Mond-Weir dual problem suggested by Egudo[7]:

(MWD) Maximize f(y)

Subject to $\tau\nabla f(y) + \lambda\nabla g(y) = 0$

$\lambda\nabla g(y) \geqq 0$

$\lambda \geqq 0$, $\tau \geqq 0$ and $\tau e = 1,$ $where$ $e = (1,......1)^T \in R^P$, Denote by Y^0 the set of all the feasible solutions of problem (MWD), i.e. ,

$\mathrm{Y}^0 = \{(y,\tau,\lambda);\ \tau\nabla f(y) + \lambda\nabla g(y) = 0\ , \lambda\nabla g(y) \geqq 0, \tau \in R^p, \lambda \in R^m, \lambda \geqq 0\}$

Theorem 4.1 (Weak duality) Suppose that

(i) $x \in X_0$ (ii) $(y,\tau,\lambda) \in \mathrm{Y}^0$ and $\tau > 0;$

(iii) Problem (VP) is strong pseudoquasi type I (φ,ρ)- univex at y with respect to some $b_0, b_1.\psi_0, \psi_1$ and ρ then $f(x) \nleq f(y).$

Proof

Suppose contrary to the result i.e, $f(x) \leq f(y).$

By the property of b_0 and ψ_0 and the above inequality, we have

$b_0(x,a)\psi_0[f(x) - f(y)] \leq 0$ (4)

By the feasibility of (y,τ,λ), we have $-\lambda^0 g(y) \leqq 0$,By the properties of b_1 and ψ_1 we get

$-b_1(x,a)\psi_1[\lambda g(y)] \leqq 0$ (5)

By the inequalities (4) and (5) and condition (iii) , we have

$\phi(x,y;(\nabla f(y),\rho)) \leq 0$ and $\phi(x,y;(\lambda\nabla g(y),\rho)) \leqq 0,$ $Since$ $\tau > 0,$ the above inequalities give $\phi(x,y;(\tau\nabla f(y) + \lambda\nabla g(y),\rho)) < 0$,which contradicts (iii). This completes the proof.

Theorem 4.2 (Weak duality) suppose that

(i) $x \in X_0$ (ii) $(y,\tau,\lambda) \in \mathrm{Y}^0$ and $\tau^0 \geq 0;$

(iii) Problem (VP) is weak strictly pseudoquasi type I (φ,ρ)- univex at y with respect to some $b_0, b_1.\psi_0, \psi_1$ and ρ then $f(x) \nleq f(y).$

Proof

Suppose contrary to the result i.e, $f(x) \leq f(y).$ By the properties of b_0 and ψ_0 and the above inequality, we get (4). By the feasibility of (y,τ,λ), and properties of b_1 and ψ_1 we get (5).

By the inequalities (4) and (5) and condition (iii) , we have

$\phi(x,y;(\nabla f(y),\rho)) < 0$ and $\phi(x,y;(\lambda\nabla g(y),\rho)) \leqq 0,$ $Since$ $\tau^0 \geq 0,$ the above inequalities give, $\varphi(x,y;(\tau^0\nabla f(y) + \lambda\nabla g(y),\rho)) < 0$,which contradicts (iii). This completes the proof .

Theorem 4.3 (Weak duality) suppose that

(i) $x \in X_0$ (ii) $(y,\tau,\lambda) \in \mathrm{Y}^0;$

(iii) Problem (VP) is weak strictly pseudo type I (φ,ρ)- univex at y with respect to some $b_0, b_1.\psi_0, \psi_1$ and ρ then $f(x) \nleq f(y).$

Proof

Suppose contrary to the result, i.e., $f(x) \leq f(y)$.By the properties of b_0, ψ_0 and the above inequality , we get (4), and the feasibility of (y,τ,λ) and properties of b_1 and ψ_1 we get (5).By the inequalities (4) and (5) and condition (iii) , we have $\phi(x, y; (\nabla f(y), \rho)) < 0$ and $\phi(x, y; (\nabla g(y), \rho)) < 0$. Which contradicts condition (iii). This completes the proof.

Theorem4.4 (Strong duality). Let z be an efficient solution for (VP) and z satisfies a constraint qualification for (VP) in Marusciac [8]. Then there exist $b \in R^p$ *and* $c \in R^m$ such that (z ,b ,c) is feasible for (MWD). If any of the weak duality in theorems 4.1-4.3 also holds. Then (z ,b ,c) is efficient solution (MWD).

Proof

Since z is efficient for (VP) and satisfies the constraint qualification for (VP), then from the Kuhn-Tucker necessary optimality condition, we obtain b>0 and $c \geqq 0$ such that $b\nabla f(z) + c\nabla g(z) = 0, cg(z) = 0$, the vector b may be normalized according to be = 1. b>0,which gives that the triple(z,b ,c) is feasible for (MWD). The efficiency of (z,b,c) for (MWD) follows from weak duality theorem. Thus completes the proof.

5. GENERAL MOND-WEIR TYPE DUALITY

In this section, we consider a general Mond-Weir type of dual problem to (VP) establish weak and strong duality theorems under some mild assumption. We consider the following general Mond-Weir type dual problem:

(GMWD) Maximize f(y)+ $\lambda_{j_0} g_{j_0}(y)e$

Subject to $\tau\nabla f(y) + \lambda\nabla g(y) = 0$ (6)

$\lambda_{j_q} g_{j_q} \geqq 0, 1 \leqq q \leqq r$ (7)

$\lambda \geqq 0$, $\tau \geqq 0$ *and* $\tau e = 1$, *where* $e = (1,......1)^T \in R^P$, $J_q, 1 \leqq q \leq r$, are partitions of the set N.

Theorem 5.1 (Weak duality) suppose that for all feasible x for (VP) and for all feasible (y, τ, λ) for (GMWD):

(a) $\tau > 0$ *and* (f $+ \lambda_{j_0} g_{j_0}(.)e, \lambda j_q g j_q(.))$ is pseudoquasi type I (φ, ρ)-univex at y for each q. $1 \leqq q \leqq r$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ ;

(b) (f $+ \lambda_{j_0} g_{j_0}(.)e, \lambda j_q g j_q(.))$ is weak strictly pseudoquasi type I (φ, ρ)-univex at y for each q. $1 \leqq q \leq r$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ ;

(c) (f $+ \lambda_{j_0} g_{j_0}(.)e, \lambda j_q g j_q(.))$ is weak strictly pseudo type I (φ, ρ)-univex at y for each q, $1 \leqq q \leqq r$ with respect to some $b_0, b_1.\psi_0, \psi_1$ *and* ρ ; then $f(x) \not\leq f(y). + \lambda j_0 g j_0(y)e.$

Proof : Suppose contrary to the result. Thus, we have

$f(x) \not\leq f(y). + \lambda j_0 g j_0(y)e.$

Since x is feasible for (VP) and $\lambda \geqq 0$, the above inequality implies that

$f(x) + \lambda J_0 g J_0(x)e. \leq f(y) + \lambda J_0 g J_0(y)e.$ (8)

By the feasibility of (y, τ, λ) inequality (7) gives

$-\lambda J_q g J_q(y) \leqq 0, 1 \leqq q \leqq r.$ (9)

Since ψ_0 *and* ψ_1 are increasing, from (8) and (9), we have

$b_0(x, y)\psi_0\{(f(x) + \lambda J_0 g J_0(x)e - f(y) + \lambda J_0 g J_0(y)e \leq 0$ (10)

$-b_1(x, y)\psi_1\{\lambda J_q g J_q(y)\} \leqq 0, 1 \leqq q \leqq r.$ (11)

By condition (a), from (10) and (11), we have

$\phi(x, y;(\nabla f(y) + \lambda J_0 gJ_0(y)e, \rho)) \le 0$

$\phi(x, y;(\lambda J_q \nabla gJ_q(y)e, \rho)) \leqq 0, 1 \leqq q \leqq r$

Since , $\tau > 0$ the above inequalities give

$$\phi(x, y;(\tau\nabla f(y) + \sum_{q=0}^{r} \lambda\nabla J_q gJ_q(y), \rho)) < 0 \qquad (12)$$

Since $J_q, 0 \leqq q \leqq r$ are partitions of the set N, (12) is equivalent to

$\phi(x, y;(\tau\nabla f(y) + \lambda\nabla g(y), \rho)) < 0$

which contradicts (6),By condition (b), from (10) and (11), we have

$\phi(x, y;(\nabla f(y) + \lambda J_0 gJ_0(y)e, \rho)) < 0,$

$\phi(x, y;(\lambda J_q \nabla gJ_q(y), \rho)) \leqq 0, 1 \leqq q \leqq r.$

Since , $\tau \ge 0,$ the above inequalities give (12), which again contradicts (6). By condition (c) ,(10) and (11), we have,
$\phi(x, y;(\nabla f(y) + \lambda J_0 gJ_0(y)e, \rho)) < 0,$ $\phi(x, y;(\lambda J_q \nabla gJ_q(y), \rho)) < 0,$ $1 \leqq q \leqq r.$ Since, $\tau \geqq 0,$ the above inequalities give (12), which again contradicts (6). This completes the proof.

Theorem 5.2 (strong duality) let z be an efficient solution for (VP) and z satisfies a constraint qualification for (VP). Then there exist $b \in R^p$ *and* $c \in R^m$ *such that* (z,b,c) is feasible for (GMWD). If any of the weak duality in theorem 5.1 holds, then (z,b,c) is an efficient solution for (GMWD).

Proof

Since z is efficient for (VP) and satisfies a generalized constraint qualification, by the Kuhn-Tucker necessary condition (see Maeda[11]),there exist b>0 and $c \geqq 0$ *such that* $b\nabla f(z) + c\nabla g(z) = 0$, $c_i g_i(z) = 0, 1 \leqq i \leqq p.$ The vector b may be normalized according to be =1, b> 0, which gives that the triplet (z,b,c) is feasible for (GMWD). The efficiency follows from the weak duality in theorem 5.1. this completes the proof.

6. CONCLUSION

In this paper, we have extended the corresponding results of Mishra [9, 5], Aghezzaf and Hachimi [1], Ferrara and Stefanescu [10] to a wider class of functions.

REFERENCES

[1] Aghezzaf B & Hachimi M, Generalized invexity and duality in multi objective programming problems, *J. Global optim*. 18(2000)91.

[2] Rueda N G, Hanson M A & Singh C, Optimality and duality with generalized convexity. *J. Optim. Theory Appl*. 86(1995) 491.

[3] Bector C R, Suneja S K & Gupta S, Univex functions and univex nonlinear programming in: Proceedings of the Administrative Sciences Association of Canada, (1992),115.

[4] Mishra S K, On multiple-objective optimization with generalized univexity, *J. Math . Anal. Appl.* 224(1998) 131.

[5] Mishra S K, Ywang S & Lai K K, Optimality and duality multiple-objective optimization under generalized type I univexity. *J. Math. Anal. Appl*. 303(2005) 315.

[6] Caristi G, Ferrara M, & Stefanescu A, Mathematical programming with (φ,ρ)-invexity in: Igor, V., Konnov, Dinh, the Luc, Alexander, M., Rubinov, (eds), Generalized convexity and Related topics, lecture notes in Economics and Mathematical system, Vol.583. Springer, 2006, 167.

[7] Egudo R R, Efficiency and generalized convex duality for multi objective programs, *J. Math. Anal. Appl*. 138(1989) 84.

[8] Marusciac I, On Fritz John Optimality criterion in multi objective optimization, *Anal. Numer , Theorie Approx*. 11(1982) 109.

[9] Mishra S K, V-invex functions and applications to multiple-objective programming problems, Ph. D. Thesis, Banaras Hindu university, Varanasi, India,1995.

[10] Ferrara M & Stefanescu M V, Optimality conditions and duality in Multi objective programming with (φ,ρ)- invexity, Yugoslav *J. Operations Research*, 18(2008) 2, 153.

[11] Maeda T, Constraint qualification in multiobjective optimization problems: differentiable case, *J. Optim. Theory Appl*. 80(1994)483.