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COMMON FIXED POINT OF SEMI COMPATIBLE MAPS IN FUZZY METRIC SPACES

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ABSTRACT
The purpose of this paper is to prove a common fixed point theorem on fuzzy metric space using the notion of semi compatibility, our result generalize the result of Som [8]. Also, we are giving an example that make strong to our result.

Keywords : Common fixed point, Fuzzy metric space, R- weakly commuting , Semi compatible maps.

AMS Subject Classification : 47H10 , 54H25**.**

INTRODUCTION

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [10], which laid the foundation of fuzzy mathematics. Kramosil and Michalek [4] introduced the concept of fuzzy metric space and modified by George and Veeramani [2]. Also Grabiec [3] has proved some fixed point results for fuzzy metric space. Sessa [6] proved some theorems of commutativity by weakening the condition to weakly commutativity. Vasuki [9] defined the R- weak commutativity of mappings of Fuzzy metric space and proved the fuzzy version of Pant's [5] theorem. Cho, Sharma and Sahu [1] introduced the concept of semi compatibility of mapps in D- metric space if condition (a) Sy = Ty implies that STy = TSy and (b) $\{Tx_n\}\to x$, $\{Sx_n\}\to x$ then $\{STx_n\}\to Tx$ as $n\to\infty$ hold. However (b) implies (a) taking $\{x_n\}\to y$ and x = Ty = Sy. So, here we define semi compatibility by condition (b) only. In this paper we used the concept of semi compatible mappings to prove further resuts.

PRELIMINARIES AND DEFINITIONS

Definitions 2.1.[7] $*: [0,1] \times [0,1] \to [0,1]$ is a continuous *t*- norm if it satisfies the following conditions :

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a \quad \forall \quad a \in [0,1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each a, b, c, d $\in [0,1]$.

Definition 2.2.[4] The triplet (X, M, $*$) is said to be Fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set on $X \times X \times [0,\infty] \rightarrow [0,1]$ satisfying the following conditions : for all x, y, z ∈ X and s, t > 0.

(FM-1) M(x, y, 0) = 0,
(FM-2) M(x, y, t) = 1 for all t > 0 if and only if x = y,
(FM-3) M(x, y, t) = M(y, x, t)
(FM-4) M(x, y, t) $*$ M(y, z, s) ≤ M(x, z, t + s),
(FM-5) M(x, y, .) : $[0,\infty] \rightarrow [0,1]$ is left continuous,
(FM-6) $\lim_{t\to\infty}$ M(x, y, t) = 1.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1.[2] Let (X, d) be a metric space. Define a $*$ b = min{a,b} and
M(x, y, t) = $\frac{t}{t+d(x,\ y)}$ for all x, y ∈ X and all t > 0. Then (X, M, $*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Lemma 2.1. [3] For all x, y ∈ X, M(x, y, .) is a non decreasing function.

Definition 2.3.[3] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, $*$) is said to be a Cauchy sequence if and only if for each ε > 0, t > 0 , there exists n_0 ∈ N such that M(x_n, x_m, t) > 1- ε for all n, m ≥ n_0.

The sequence $\{x_n\}$is said to converge to a point x in X if and only if for each ε > 0, t > 0 , there exists n_0 ∈ N such that M(x_n, x, t) > 1- ε for all n ≥ n_0.

A Fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence in it converge to a point in it.

Definition 2.4.[5] Two self maps A and S of Fuzzy metric space (X, M, $*$) are said to be weakly commuting if

M(ASx, SAx, t) ≥ M(Ax, Sx, t) for every x ∈ X.

The notion of weak commutativity is extended to R-weak commutativity by Vasuki [9] as

Definition 2.5.[9] Two self maps A and S of Fuzzy metric space (X, M, $*$) are said to be R-weakly commuting provided there exist some positive real number R such that

M(ASx, SAx, t) ≥ M(Ax, Sx, $\frac{t}{R}$) for all x ∈ X.

The weak commutativity implies R-weak commutativity and converse is true for R ≤ 1.

Definition 2.6. A pair (A, S) of self mappings of a Fuzzy metric space is said to be Semi compatible if M(ASx_n, Sx, t)→1 for all t > 0 whenever $\{x_n\}$ is a sequence in X such that Ax_n, Sx_n → p for some p in X as n→∞.

It follows that (A, S) is Semi compatible and Ay = Sy imply ASy = SAy by taking $\{x_n\}$= y and x = Ay = Sy.

Remark 2.1. Let (A,S) be a pair of self mappings of a Fuzzy metric space (X, M, $*$). Then (A,S) is R-weakly commuting implies (A, S) is Semi compatible but the converse is not true.

Using R-weak commutativity, Som [8] proved some results. Here we generalized the result of Som [8] by replacing the assumption of R-weakly commuting maps to Semi compatible maps.

Example 2.2. Let X = [0, 2] and $a * b = \min\{a, b\}$. Let $M(x, y, t) = \frac{t}{t + d(x, y)}$ be the standard Fuzzy metric space induced by d, where $d(x, y) = |x - y|$ for all x, y ∈ X, define

$$A(x) = \begin{cases} 2, & x \in [0,1] \\ \frac{x}{2}, & x \in (1,2] \end{cases} \qquad S(x) = \begin{cases} 1, & x \in [0,1) \\ 2, & x = 1 \\ \frac{x+3}{5}, & x \in (1,2] \end{cases}$$

Now for $1 < x \leq 2$ we have

$$Ax = \frac{x}{2}, \quad Sx = \frac{x+3}{5} \quad \text{and} \quad ASx = \frac{x+3}{10}, \quad SAx = \frac{x+6}{10}$$

then $M(ASx, SAx, t) = \frac{10t}{10t + 3}$

$$M(Ax, Sx, \tfrac{t}{R}) = \frac{10t}{10t + 3(2-x)R}.$$

We observe that $M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})$ which gives $R \geq \frac{1}{(2-x)}$

Therefore we get there no R for x ∈ (1, 2] in X.

Hence (A,S) is not R-weakly commuting.

Now we have S(1) = 2 = A(1), and S(2) = 1 = A(2)

also SA(1) = AS(1) and AS(2) = 2 = AS(2)

Let $x_n = 2 - \frac{1}{2n}$

Hence $Ax_n \to 1$, $Sx_n \to 1$ and $ASx_n \to 2$

Therefore $M(ASx_n, Sy, t) = (2, 2, t) = 1$.

Hence (A, S) is Semi compatible but not R-weakly commuting.

MAIN RESULTS

Theorem 3.1. Let S and T be two continuous self mappings of a complete Fuzzy metric space $(X, M, *)$ such that $a * b = \min(a, b)$ for all a, b in X. Let A be a self mapping of X satisfying the following conditions:

(1) $A(X) \subset S(X) \cap T(X)$,

(2) (A,S) and (A,T) are semi compatible,

(3) $M(Ax, Ay, t) \geq r \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, Ay, t), M(Ty, Ay, t)\}$
for all x, y ∈ X and t > 0, where $r : [0, 1] \to [0, 1]$ is a continuous function such that

(4) $r(t) > t$, for each $0 < t < 1$.

Then A, S, T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any arbitrary point.

Since $A(X) \subset S(X)$ then there must exists a point $x_1 \in X$ such that $Ax_0 = Sx_1$.

Also, since $A(X) \subset T(X)$, there exists another point $x_2 \in X$ such that $Ax_1 = Tx_2$.

In general, we get a sequence $\{y_n\}$ recursively as

$$y_{2n} = Sx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Tx_{2n+2} = Ax_{2n+1},\ n \in N \cup \{0\}.$$

Let $M_{2n} = M(y_{2n+1}, y_{2n}, t) = M(Ax_{2n+1}, Ax_{2n}, t)$. Then, $M(Ax_{2n+2}, Ax_{2n+1}, t) = M_{2n+1}$.

Using inequality (3), we get

$$M_{2n+1} \geq r \min\{M(Sx_{2n+2}, Tx_{2n+1}, t), M(Sx_{2n+2}, Ax_{2n+2}, t), M(Sx_{2n+2}, Ax_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n+1}, t)\}$$

$$= r \min\{M(Ax_{2n+1}, Ax_{2n}, t), M(Ax_{2n+1}, Ax_{2n+2}, t), M(Ax_{2n+1}, Ax_{2n+1}, t), M(Ax_{2n}, Ax_{2n+1}, t)\}$$

$$= r \min(M_{2n}, M_{2n+1}, M_{2n}) \tag{3.1}$$

If $M_{2n} > M_{2n+1}$, then by definition of r we have

$M_{2n+1} \geq r(M_{2n+1}) > M_{2n+1}$, a contradiction. So, $M_{2n+1} \geq M_{2n}$.

Thus, from (3.1), we get $M_{2n+1} \geq r(M_{2n}) \geq M_{2n}$. (3.2)

Hence $\{M_{2n}\}$ where $0 \leq n \leq \infty$ is an increasing sequence of positive numbers in [0, 1] and therefore, tends to a limit $L \leq 1$.

We claim that L = 1. If L < 1, then on taking limit $n \to \infty$ in (3.2), we get

$$L \geq r(L) \geq L;$$

i.e. $r(L) = L$, which contradicts the fact that L < 1.

Hence, L = 1.

Now for any positive integer p,

$$M(Ax_n, Ax_{n+p}, t) \geq M(Ax_n, Ax_{n+1}, \tfrac{t}{p}) * M(Ax_{n+1}, Ax_{n+2}, , \tfrac{t}{p}) * \ldots\ldots\ldots * M(Ax_{n+p-1}, Ax_{n+p}, ,\tfrac{t}{p})$$

$$> (1 - \varepsilon) * (1 - \varepsilon) * \ldots * (1 - \varepsilon) \text{ (p-times)} = 1 - \varepsilon.$$

Thus, $M(Ax_n, Ax_{n+p}, t) > 1 - \varepsilon,\ \forall\, t > 0$.

Hence $\{Ax_n\}$ is a Cauchy sequence in X. Since X is complete $\{Ax_n\} \to z \in X$. Hence the subsequences $\{Sx_n\}$ and $\{Tx_n\}$ of $\{Ax_n\}$ also tends to the same limit.

Case I. Since S is continuous. In this case we have

$SAx_n \to Sz$, $SSx_n \to Sz$

Also (A, S) is semi compatible, we have $ASx_n \to Sz$

Step I. Let $x = Sx_n$, $y = x_n$ in (3) we get

$$M(ASx_n, A x_n, t)) \geq r \min\{M(SSx_n, Tx_n, t), M SSx_n, ASx_n, t), M(SSx_n, Ax_n, t), M(Tx_n, Ax_n, t)\}.$$

Taking limit as $n \to \infty$,

$$M(Sz, z, t) \geq r \min\{M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, z, t)\}.$$

$$\geq r M(Sz, z, t),$$

$$> M(Sz, z, t).$$

So, we get $Sz = z$.

Step II. By putting $x = z$, $y = x_n$ we get $Az = z$.

Hence , $Az = z = Sz$.

Case II. Since T is continuous. In this case we have $TTx_n \to Tz$, $TAx_n \to Tz$.

also (A, T) is semi compatible $ATx_n \to Tz$.

Step I. Let $x = x_n$, $y = Tx_n$ in (3) we get

$$M(Ax_n, ATx_n, t) \geq r \operatorname{Min}\{M(Sx_n, TTx_n, t), M(Sx_n, Ax_n, t), M(Sx_n, ATx_n, t), M(TTx_n, ATx_n, t)\}$$

$$M(z, Tz, t) \geq r \min\{M(z, Tz, t), M(z, z, t), M(z, Tz, t), M(Tz, Tz, t)\}.$$

$$\geq r M(z, Tz, t),$$

$$> M(z, Tz, t).$$

So, we get Tz = z. Thus, we have Az = Sz = Tz = z.

Hence z is a common fixed point of A, S and T.

Uniqueness : Let u be another common fixed point of A, S and T, Then

$$Au = Su = Tu = u.$$

Put x = z, y = u in (3), we get

$$M(Az, Au, t)) \geq r \min\{M(Sz, Tu, t), M (Sz, Az, t), M(Sz, Au, t), M(Tu, Au, t)\}.$$

Therefore

$$M(z, u, t)) \geq r \min\{M(z, u, t), M (z, z, t), M(z, u, t), M(u, u, t)\}.$$

$$\geq r \min\{M(z, u, t), 1, M(z, u, t),1 \}.$$

$$\geq r\, M(z, u, t),$$

$$> M(z, u, t)$$

which gives z = u.

Therefore z is a unique common fixed point of A, S and T.
If we take T = S then we get following corollary

Corollary 3.2. let S be a continuous mapping of a complete Fuzzy metric space $(X, M, *)$ such that $a * b = \min (a, b)$ for all a, b in X. Let A be a self mapping of X satisfying the following conditions:

(1) $A(X) \subset S(X)$,
(2) (A, S) is semi compatible,
(3) $M(Ax, Sy, t) \geq r \min\{M(Sx, Sy, t), M(Sx, Ax, t), M(Sx, Ay, t), M (Sy, Ay, t)\}$
for all x, $y \in X$ and $t > 0$, where $r : [0, 1] \to [0, 1]$ is a continuous function such that
(4) $r(t) > t$, for each $0 < t < 1$.

Then A and S have a common fixed point in X.

Theorem 3.2. Let S and T be two continuous self mappings of a complete Fuzzy metric space $(X, M, *)$ such that $a * b = \min (a, b)$ for all a, b in X. Let A and B be two self mappings of X satisfying the following conditions:

(1) $A(X) \cup B(X) \subset S(X) \cap T(X)$,
(2) (A,T) and (B, S) are semi compatible pairs,
(3) $aM (Tx, Sy, t) + bM(Tx, Ax, t) + c\, M(Sy, By, t)$
$+ \max\{M(Ax, Sy, t), M(By, Tx, t)\} \leq q\, M(Ax, By, t)$

for all x, y ∈ X, where a, b, c ≥ 0 with q < (a + b + c) < 1.
Then A,B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any arbitrary point.

Since A(X) ⊂ S(X) then there must exists a point $x_1 \in X$ such that $Ax_0 = Sx_1$.

Also since A(X) ⊂ T(X), there exists another point $x_2 \in X$ such that $Ax_1 = Tx_2$.

In general, we get a sequence $\{y_n\}$ recursively as

$$y_{2n} = Sx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Tx_{2n+2} = Ax_{2n+1},\ n \in N \cup \{0\}.$$

Using inequality (3), we get similarly as som [9] that for $\frac{a+b}{q-c} > 1$ a Cauchy sequence in X. Hence, the sequence $\{Ax_{2n,}\}$,$\{Bx_{2n+1}\}$,$\{Sx_{2n+1}\}$ and $\{Tx_{2n+2}\}$ are Cauchy and converge to same limit, say z.

Case I. Since T is continuous. In this case we have

$$TAx_n \to Tz\ , \quad TTx_n \to Tz$$

Also (A, T) is semi compatible, we have $ATx_n \to Tz$

Step I. Let $x = Tx_n$, $y = x_n$ in (3), we get

$$aM(TTx_n, Sx_n, t) + bM(Tx_n, ATx_n, t) + c\,M(Sx_n, Bx_n, t)$$
$$+ \max\{M(ATx_n, Sx_n, t), M(Bx_n, TTx_n, t)\} \leq qM(ATx_n, Bx_n, t)$$

Taking limit as $n \to \infty$, we get

$$aM(Tz, z, t) + bM(z, Tz, t) + c\,M(z, z, t)$$
$$+ \max\{M(Tz, z, t), M(z,Tz, t) \leq qM(Tz, z, t)$$

i.e., $aM(Tz, z, t) + bM(z, Tz, t) + c + M(Tz, z, t) \leq qM(Tz, z, t)$

i.e., $c \leq (q - a - b - 1)\,M(Tz, z, t)$

i.e., $M(Tz, z, t) \geq \frac{c}{q-a-b-1} > 1$

which gives Tz = z.

Step II. Putting $x = z$ and $y = x_n$ in (3) we get

$$aM(Tz, Sx_n, t) + bM(Tz, Az, t) + cM(Sx_n, Bx_n, t) + \max\{M(Az, Sx_n, t), M(Bx_n, Tz, t)\} \leq qM(Az, Bx_n, t)$$

Taking limit as $n \to \infty$, we get

$$aM(z, z, t) + bM(z, Az, t) + cM(z, z, t) + \max\{M(Az, z, t), M(z, z, t)\} \leq qM(Az, z, t)$$

i.e. $$a + bM(z, Az, t) + c + \max\{M(Az, z, t), 1\} \leq qM(Az, z, t)$$

i.e. $$a + c + 1 \leq (q - b) M(Az, z, t)$$

i.e $$M(Az, z, t) \geq \frac{a+c+1}{q-b} > 1$$

which gives Az = z.

Hence, Az = z = Tz.

Case II. Similarly since S is continuous and (B, S) is semi compatible
we get Bz = z = Sz.
Thus we have Az = Bz = Tz = Sz = z.

Hence z is a common fixed point of A, B, S and T, and easily we can prove that it is a unique common fixed point of A, B, S and T.

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