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REMARKS ON CERTAIN SELECTED FIXED POINT THEOREMS - II

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ABSTRACT
Recent common fixed point theorems due to Kumar et al [8] and Jungck [11] are used to derive two common fixed point theorems for four finite families of mappings in complete and compact metric spaces respectively.

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Let R^+ denotes the set of non-negative real numbers and Y be the family of mappings f from R^+ into R^+ such that

(i) f is non-decreasing

(ii) f is upper semi-continuous in each coordinate variables

(iii) $g(t) = f(t, t, a_1t, a_2t, t) < t$ where $g : R^+ \to R^+$ is a mapping with $g(0) = 0$ and $a_1 + a_2 = 2$.

We need to recall the following:

Definition 1 (Jungck and Rhoades [12])
Two maps $A, S : X \to X$ are said to be coincidentally commuting if they commute at their coincidence points.

The following fixed point theorem appears in Kumar et al [8].

Theorem 2 Let F, G, A and B be self-mappings of a metric space (X, d) with $A(X) \subset G(X)$ and $B(X) \subset F(X)$

(a) $[1 + p\, d(Fx, Gy)]d(Ax, By) < p \max\{d(Fx, Ax).d(Gy, By), d(Fx, By)d(Gy, Ax)\}$
$$+ F(d(Fx, Gy), d(Fx, Ax), d(Gy, By), d(Fx, By), d(Gy, Ax)) \quad (2.1)$$
for all x, y in X, FÎy with $p > 0$.

If one of A(X), B(X), G(X), F(X) is complete subspace of X, then

(b) (A, F) has a coincidence point.

(c) (B, G) has a coincidence point.

Further if the pairs (A, F) and (B, G) are coincidentally commuting, then A, B, F and G have a unique common fixed point z which also remains the unique common fixed point of both the pairs separately.

As an application of Theorem 2, we derive a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 3 Let $\{S_1, S_2, \ldots, S_m\}$, $\{T_1, T_2, \ldots, T_n\}$, $\{I_1, I_2, \ldots, I_p\}$and $\{J_1, J_2, \ldots, J_q\}$ be four finite families of self - mappings of a metric space (X, d) with $S = S_1S_2 \ldots S_m$, $T = T_1T_2 \ldots Tn$, $I = I_1 I_2 \ldots I_p$ and $J = J_1 J_2 \ldots J_q$ satisfying the following conditions:
(d) $S(X) \subset J(X)$, $T(X) \subset I(X)$
(e) one of S(X), T(X), I(X) and J(X) is a complete subspace of X
(f) $[1+ p\, d(Fx,Gy)]\, d(Ax, By) < p \max\{d(Fx, Ax).d(Gy, By), d(Fx, By).d(Gy, Ax)\} + F(d(Fx, Gy), d(Fx, Ax), d(Gy, By), d(Fx, By), d(Gy, Ax))$.

Then

(g) (S, I) have a point of coincidence
(h) (T, J) have a point of coincidence.

Moreover, if $S_i S_j = S_j S_i$, $I_k I_l = I_l I_k$, $T_r T_s = T_s T_r$, $J_t J_u = J_u J_t$, $S_i I_k = I_k S_i$, and $T_r J_t = J_t T_r$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1,2, \ldots, n\}$ and t, $u \in I_4 = \{1, 2, \ldots, q\}$. Then (for all $i \in I$, $k \in I_2$, $r \in I_3$ and $t \in I_4$) S_i, I_k, T_r and J_t have a common fixed point.

Proof The conclusions (g) and (h) are immediate as S, T, I and J satisfy all the conditions of Theorem 2. Now appealing to componentwise commutativity of various pairs, one can immediately prove that SI = IS and TJ = JT and hence obviously both the pairs (S, I) and (T, J) are coincidentally commuting. Note that all the conditions of Theorem 2. (for mappings S, T, I and J) are satisfied ensuring the existence of unique common fixed point z. Now we need to show that z remains the fixed point of all component maps. For this consider

$$S(S_iz) = ((S_1,S_2, \ldots,)S_i)z = (S_1S_2 \ldots S_{m-1})((S_m S_i)z) = (S_1 \ldots S_{m-1})(S_iS_mz)$$
$$= (S_1 \ldots S_{m-2})(S_{m-1}S_i(S_mz)) = (S_1 \ldots)S_i(S_{m-1}(S_mz))$$
$$= (S_1 \ldots S_{m-4}(S_m - 3S_i)((S_{m-2}(S_mz))) = \ldots = (S_i(S_1S_2 \ldots S)z = S_i(Sz) = S_iz.$$

Similarly one can show that
$S(I_k z) = I_k (Sz) = I_kz$, $I(I_kz) = I_k(Iz) = I_kz$. $I (S_iz) = S_i (Iz) = S_iz$.
$T(T_rz) = T_r (Tz) = T_rz$. $T (J_tz) = J_t (Tz) = J_tz$. $J(T_rz) = T_r (Jz) = T_rz$. $J (J_tz) = J_t (Jz) = J_tz$
which show that (for all i, r, k and t) S_iz and I_kz are other fixed points of the pair (S, I) whereas T_rz and J_iz are the other fixed points of the pair (T, J). Now appealing to the uniqueness of common fixed points of the pairs (S, I) and (T, J) one gets (for all i, r, k and t) $z = S_iz = T_rz = I_kz = J_tz$ which show that z is a common fixed point of S_i, T_r, I_k and J_t for all i, r, k and t.

Remark 4
By setting $S_1 = S_2 = \ldots = S_m = A$, $T_1 = T_2 = \ldots = T_n = B$, $I_1 = I_2 = \ldots = I_k = F$ and $J_1 = J_2 = \ldots = J_t = G$, we get a fixed point theorem for A_m, B_n, F_k and G_t which generalizes the Theorem 2 due to Kumar *et al.* [8]. In process several known results are

generalized and improved (e.g. Theorem 2.3 [14]). More particularly by setting m = n = p = q and (for all i, r, k and t) $S_i = T_r = I_k = J_t = F$, we deduce a fixed point theorem for an iterates of F which presents a generalization to the theorem of Bryant [1].

Remark 5 By choosing f suitably one can derive improved versions of a multitude of relevant known common fixed point theorems involving four mappings especially those contained in Lal *et al.* [15], Singh-Meade [17], Husain - Sehgal [6], Khan - Imdad [14],

Jungck [9], Ciric [2], Singh - Singh [18], Fisher [4, 5], Das - Naik [3], Kannan [13], Rhoades [16] and several others. Also setting p = 0 and choosing S, T, I, J and f suitably one can deduce the results proved in the above cited references and many others. Next we wish to indicate a similar result in compact metric spaces. For this purpose one can adopt a general fixed point theorem for commuting mappings in compact metric spaces due to Jungck [11], which was originally proved for compatible mappings (a notion due to Jungck [10]).

Theorem 6 ([11])

Let A, S, I and J be self-mapping of a compact metric space (X, d) with $A(X) \subset J(X)$ and $S(X) \subset I(X)$. If the pairs (A, I) and (S, J) are commuting and

$d(Ax, Sy) < M(x, y)$, for all x, y Î X

where $M(x, y) = \max \{d(Ix, Jy), d(Ix, Ax), d(Jy, Sy), [d(Ix, Sy) + d(Jy, Ax)]/2\}$with $M(x,y) > 0$, then A, S, I and J have a unique common fixed point provided all four mappings A, S, I and J are continuous.

As an application of Theorem 6 one can have the following:

Theorem 7

Let $\{S_1, S_2, \ldots, S_m\}$, $\{T_1, T_2, \ldots, T_n\}$, $\{I_1, I_2, \ldots, I_p\}$and $\{J_1, J_2, \ldots, J_q\}$ be four finite families of self - mappings of a complete metric space (X, d) with $S = S_1S_2 \ldots S_m$, $T = T_1T_2 \ldots T_n$, $I = I_1I_2 \ldots$ Ip and $J = J_1J_2 \ldots J_q$ continuous mappings satisfying (d) and $d(Sx, Ty) < M(x, y)$ for all $x, y \in X$ where $M(x, y) = \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), 1/2[d(Ix, Ty) + d(Jy, Sx)]\}$with $M(x, y) > 0$. Moreover if $S_i S_j = S_j S_i$, $I_k I_l = I_l I_k$, $T_r T_s = T_s T_r$, $J_t J_u = J_u J_t$, $S_i I_k = I_k S_i$, and $T_r J_t = J_t T_r$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1, 2, \ldots, n\}$ and $t, u \in I_4 = \{1, 2, \ldots, q\}$, Then (for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$) S_i, I_k, T_r and J_t have a unique common fixed point.

Proof The proof is essentially the same as that of Theorem 3, hence it is omitted.

Remark 8 By setting $S = S_1 S_2$, $T = T_1T_2$, $I = I_1$ and $J = J_1$ in Theorem 3 and Theorem 7 one can deduce Theorem 1.2 and Theorem 1.4 of Imdad [7] respectively.

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