

## **New method of contact detection in experimental granular shear flow**

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### **ABSTRACT**

Different assemblies of granular materials are generated with varying solid fractions and are sheared in the 2D experimental shear flow apparatus with the objective of developing efficient contact detection methodology. The flow is recorded by high speed video camera, and the spatial positions of particle centers are obtained through image processing techniques. The experimental setup was managed to inspect the collision, collision time and collision partners while the experiment is on. Important observations of pre- and post- collision phenomena were made in order to find the precise criteria for distinguishing contacts. Based on these observations together with simple rules of physics, new contact detection algorithm is proposed. The results obtained from proposed method are compared with the visual observations in the digital video of the physical experiment with the help of motion analytical and particle tracking PTV software. It is revealed that the proposed method could detect the contacts with more than 90 percent accuracy in an average and is considerably accurate than the existing methods of contact detection in the physical experiments.

**Keywords:** Velocity vector, contact detection, collision, granular materials, contact duration

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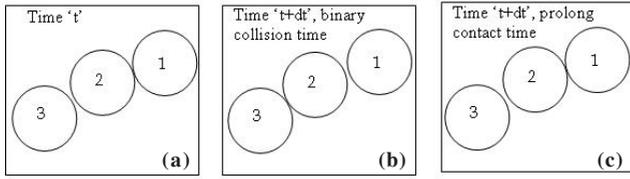
### **INTRODUCTION**

A granular material is an assembly of a large number of discrete solid particles with different sizes. Cereal grains, sands, gravels, rock fragments, coal, fertilizers and chemicals in powder form are some of the examples of granular materials. Because of its wide application area, large amount of these materials are moved, stored and processed every day. Many of them are transported by natural means and many others are transported artificially for some purpose. Granular materials are the concern of many interdisciplinary field of research including fluid mechanics, geophysics, civil engineering, agriculture and chemical engineering. They are also important for geological processes where landslides, soil erosion and sedimentations determine much of the morphology of the Earth. Since the granular materials are discrete, highly discontinuous and inhomogeneous, the deformation of the bulk of the granular materials is accomplished due to the displacement of its constituent particles relative to each other. Inter-particle contacts are responsible for force transmission throughout the granular mass. Therefore, the methods that deal with the bulk of the solid materials like continuum theories for granular materials (Cowin and Goodman 1976; Cowin 1974; Goodman and Cowin 1972) are not truly applicable in characterizing the granular assembly.

Particle scale contact information is necessary to understand the granular flow and its mechanics. Due to the difficulty on getting the particle scale information, accurate detection of contacts is still a major challenge in the physical

experiments. Its scope is quite high when each of the particle-particle collisions is to be distinguished so that binary and multi-particle interactions can be separated. This is essential because when the particle concentration is moderate to high. It is natural to expect that more than two particles interact at a time and that the interaction continues for the time greater than instantaneous binary collision time. It is important to note that instantaneous binary collision is the major assumption undertaken in most of the theoretical or numerical simulation studies (Bagnold 1954; Savage and Jefferey 1981). Therefore, such models are to be validated from the physical experiments, since the existing theoretical models may give misleading results especially when the particle concentration is not much less. Turning point for such validation is getting accurate method of particle-particle interaction in the physical experiment. We can illustrate why the accurate method of contact detection is essential from a simple example. Let us consider the assembly of three particles where particles 1 and 2 make a collision at time frame  $t$  (Fig. 1a). If we consider the instantaneous collision (binary collision) as is considered in Bagnold's approach (or kinetic theory models), particles 1 and 2 will be no more in contact at next time frame  $t + \Delta t$ , rather it is quite probable that particle 2 will make collision with yet another particle 3 at this time frame (Fig. 1b), increasing the collision rate.

However, if we consider the collision time is greater than binary (multiple collision), particles 1 and 2 will remain in contact even in the time frame (Fig. 1c) and there will be less probability of particle 2 colliding with particle 3 at this time. If we consider similar example in the assembly of many



**Fig. 1: Schematic diagram to show the particle-particle interaction modes (particle movement direction is from right to left): (a) Status of particle interaction at time frame  $t$ . (b) Possible status of particle interaction at next time step  $t + \Delta t$  if we consider binary collision. (c) Possible status of particle interaction at time step if we consider the interaction time greater than binary.**

particles, it can be expected that there will be clearly different mechanism than what is assumed in these existing models because of the interactions between group of particles rather than interaction between individual particles.

Collision detection models that use only the distance between particles (Vemuri et al. 1998; Allen and Tildesley 1987; Hockney and Eastwood 1981) to find the collisions, can not be accurate as there are lots of noises and limitations of precision in the physical experiments. Detail review of the methods that use the spatial subdivisions of the assembly to find the contacts which are especially useful in the numerical simulations can be found in Vemuri et al. (1998). Since the theoretical and numerical simulations are to be validated from the physical experiments, it is imperative to find the better and efficient method of contact detection that are useful in the physical experiments. Different researchers have used the image processing techniques so as to find the positions and velocities of particles by direct imaging the experimental run and tracking their movement (Mahmood et al. 2009; You et al. 2004; Blair and Kudrolli 2003; Elliot et al. 1998). In this study, the image processing technique is used for particle tracking. In order to make more promising contact detection methods, careful investigation of the collision behavior in the experiment is more important. With this objective, the granular assembly of idealized discs is sheared in the newly designed 2D shear flow apparatus and the experimental runs are captured with the help of high speed video camera. The digital video is later analyzed by motion analytical and particle tracking software to get the spatial position data and the trajectories. Using this setup, it is possible even to detect the time and partner of collision so as to observe the changes in particle movement after collision and also to validate the contact detection algorithm. In this study, it is observed that in general when particles make collision, their velocity direction changes significantly. This criterion was adopted also by Blair and Kudrolli (2003) in addition to the distance between particles. However, while tracking particle movement, it is observed that if particles collide, sometimes the velocity magnitude changes significantly without causing a significant change in the velocity direction.

Therefore, if only the velocity direction change criterion is used, many of the contacts can be easily missed. Moreover, the Blair and Kudrolli (2003) method is formulated only to find the contacts but it does not have any method to look what if the particles remain continuously in contact for long time. This is highly probable in more dense granular flow. Moreover, it is noticed that if the collision is not instantaneous, the relative velocity of the pairs can precisely give idea about whether the contact is lost or the particles are in continuous contact rather than that by the velocity vector change. Mahmood et al. (2009) introduced the velocity magnitude change criteria as well in addition to the velocity direction, however, they did not check whether the pairs are in continuous contact or separate after making the contacts. Looking at these limitations of the previous methods of contact detection and its importance in the contact dynamics studies, the major objective of this study is to formulate more accurate contact detection method so as it can be used to study the contact dynamics in more precise way. The proposed method will be validated from the experimental results.

## EXPERIMENT

### Apparatus

The shear flow apparatus is newly designed in Saitama University, Japan which consists of an inner movable wall and an outer stationary wall that rest on a bottom horizontal plate (Fig. 2). The inner and outer walls, both are roughened by glued-on circular flanges of about the same radius as the particles to enhance the shear flow. The outer wall is placed on a horizontal steel plate atop several ball bearings to minimize friction between them. The inner wall velocity can be freely controlled by changing the speed of a DC motor joined to it. The spacing between the outer and inner walls, called the shear cell height, is 11.5 cm. The distance between the upper and lower plates is 8 mm, which is slightly larger than the particle thickness (6 mm). Apparatus details and physical properties of particles used in the experiments are given in Tables 1 and 2, respectively. The apparatus is covered on top with an acrylic plate to keep the particles from escaping, to generate stable flow, and to capture digital images during the experiment.

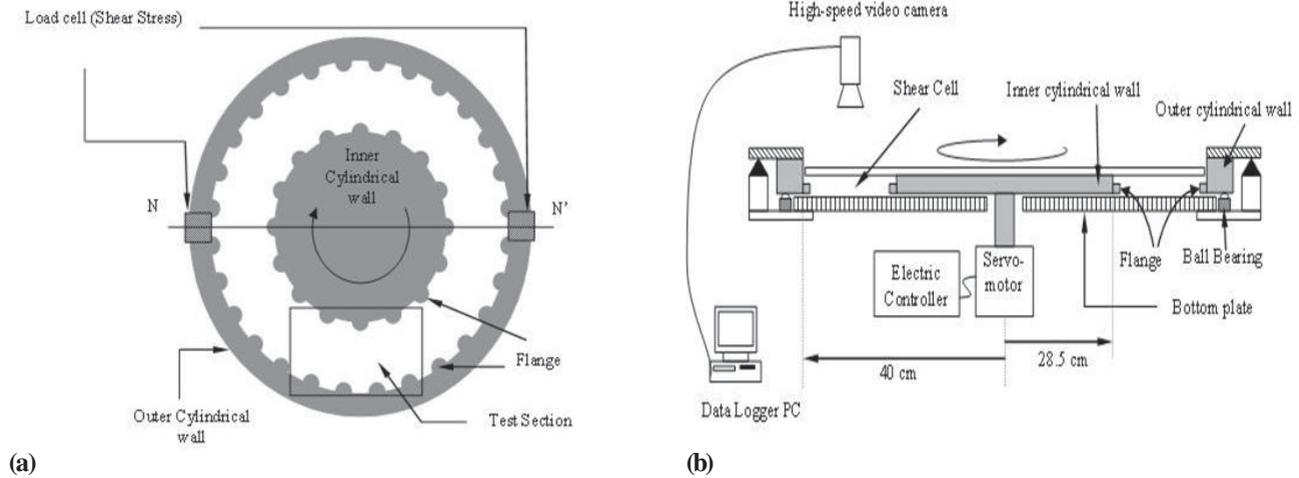
Uniformly sized plastic discs with diameter of 2.3 cm and marked at the center were used in the experiment to permit evaluating spatial coordinates of the particle. The size of the marks is kept optimum so that they are distinct but not necessarily large. The ratio of the width of the mark to the particle diameter is kept nearly 0.08. The coefficient of friction between the particles and the shear cell floor is about 0.30. The shear rate,  $\dot{\gamma} = v_w / H$ , (defined as the inner wall velocity  $v_w$  divided by the shear cell height  $H$ ), is changed by varying the velocity of the inner wall. The shear flow is given ample time to come to a stable state, which takes about 2 minutes after initiation of shearing. The state is judged stable when every particle has become part of the shear flow, and the flow is homogeneous without stoppage at any moment. After

**Table 1: Details of experimental shear cell apparatus**

Parameter	Value
Motor speed	3~180 rpm
Maximum torque of motor	15 N-m
Radius of inner wall	28.5 cm
Radius of outer wall	40 cm
Diameter of flange	2.47 cm
No. of flanges attached to inner wall	37
No. of flanges attached to outer wall	52

**Table 3: Capacity of high speed video camera**

Parameter	Value
Frame rate	60~1,000 Hz
Shutter speed	30~10,000 Hz
Pixel resolution	1,280 x 512
Number of frames possible	2048
Size of the test section	20 x 11.5 cm <sup>2</sup>
Number of pixels in one particle	41.4
Type of color	Monochrome



**Fig. 2: Schematic diagram of shear flow apparatus: (a) Plan view, and (b) Cross section**

establishing the proper arrangement of the light system, the stable shear flow in the test section of the shear cell is recorded with the high speed video camera (FASTCAM X 1280 PCI), which is connected to a frame grabber. The detailed specifications of the high speed video camera are given in Table 3. Each experimental run consisted of 2048 digital video frames at 1000 frames per second and a resolution of 1280X512 pixels. The recorded digital video is later transferred to the particle tracking software to determine particles' spatial positions.

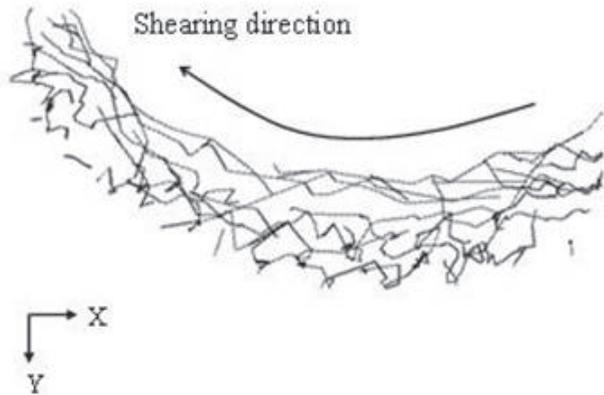
### Particle tracking

The digital video of each experimental run captured by the FASTCAM-X 1280 PCI is transferred to the Dipp-Motion 2D motion analysis and particle tracking software produced by DITECT Co. Ltd., Japan. The particle tracking software determines the coordinates of the particles' centers. Particle tracking software recognizes a black circular geometric shape in the image and looks for the closest matching shape to track in the following time steps. To achieve the maximum possible contrast between the floor of the shear cell and the particles, white colored plastic discs are used. The black circular marks in the center of the particles create the highest possible contrast, and the software can follow the circular mark even if the particle undergoes rotation.

**Table 2: Physical properties of plastic disks**

Parameter	Value
Diameter, D (mm)	23
Thickness (mm)	6
Mass (g)	2.306
Density (g/cm <sup>3</sup> )	0.934
Coefficient of restitution, e	0.87
Particle-particle coefficient of friction	0.265
Particle-floor coefficient of friction	0.3
Young's modulus (MPa)	980
Poisson's ratio	0.2
Tensile strength (MPa)	44.1

The position-tracking methodology records the coordinates  $(x_{1i}, y_{1i})$  of the  $i^{\text{th}}$  particle from frame 1 and maps them to frame 2, where their new coordinates are recorded as  $(x_{2i}, y_{2i})$ , and so on. After tracking the positions of this particle for all the time frames, the next particle is tracked in a similar fashion. After making necessary adjustments in the software and applying a conversion factor to calculate physical coordinates from pixels as given in Table 3, the spatial

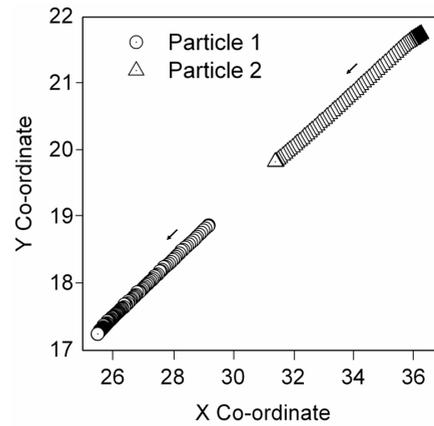


**Fig. 3: Typical particle movement trajectories in shear cell (solid fraction of 0.40 and shear rate of 34.6/s)**

coordinates of the center of each particle at each frame are obtained. Other parameters, including the velocity vectors and contact properties, were determined using these spatial coordinates. To judge the accuracy of tracked data, experiments were conducted using particles that are marked at two places equidistant from the particle center and whose spatial positions are tracked. Ideally, if the data is hundred percent accurate, the distance between two mark points should be same in each of the frame owing the fluctuation distance to be zero. It is found that the distance between two marked points calculated at different time frames using the position data from the particle-tracking method is greater than the actual distance by 1 mm in an average, which is about 4% of the particle diameter. It is therefore assumed that the spatial position data have not serious limitations in terms of accuracy.

### OBSERVATIONS IN THE EXPERIMENT

To find the important pre- and post- collision phenomena of the inter-particle collisions several test experiments were run and keenly investigated in the motion analytical and particle tracking software. It is observed that after the particles undergo collision, mostly the particles suffer significant velocity direction change compared to previous time step. The twist and turns seen in trajectories of particles (Fig. 3) are identified to be the result of collision. One of the important observations is that when one of the almost stationary particles is hit by the other from the back, there can be no change in velocity direction even after collision (Fig. 4). This is especially possible when the particle concentration is very low so that particles have enough space to travel without any disturbance. However, in this case, the momentum of one particle is transferred to the other during collision; therefore, one of the particles got increase in the velocity magnitude while the other almost stops (Fig. 5). On travelling to hit the stationary particle, the moving particle also undergo collision with some of the other particles in the assembly.



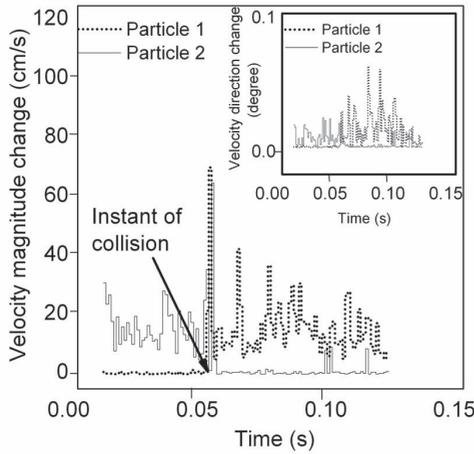
**Fig. 4: Locus of two particles where moving Particle 2 hits the stationary Particle 1 from back (very sparse case), arrow indicates the particle movement direction.**

Therefore, there are fluctuations in the velocity magnitude of particle 1 before collision and particle 2 after collision in Fig. 5.

If both of the particles are moving in the same direction and one of them hit the other from back, both of the particles could experience change in velocity magnitude: increase in one and decrease in the other. If however, both of the particles are moving towards each other, after the collision both of them experience significant change in the velocity direction (Fig. 6) in addition to the change in velocity magnitude. It is very important observation to find the potential partners of collision. The distance between the particle centers can then be used to distinguish the collision partners. The next question is what happened if both of the particles collided but they remain in contact for some more time. This can be solved by considering the relative velocity of this pair along the contact normal direction. If the particles remain in contact, they will move together and therefore, the relative velocity will be zero. If the particles are not much rigid and there is possibility of overlapping, then the relative velocity along the contact normal direction will be negative. The overlapped particles will separate once the relative velocity becomes more than zero. The duration of contact can thus be calculated by subtracting the time first collision from the time of separation. In this way criteria for detecting the contacts, and finding the contact durations can be formulated.

### COLLISION-DETECTION ALGORITHM

The collision-detection algorithm adopted in this study has three major components: finding potential candidates for collision, detecting collision partners, and checking if contact is lost or continued. The flow chart for detecting collisions is shown in Fig. 7. The flow chart for determining whether a collision vanishes or continues in the next time steps is shown in Fig. 8, which is also part of the contact

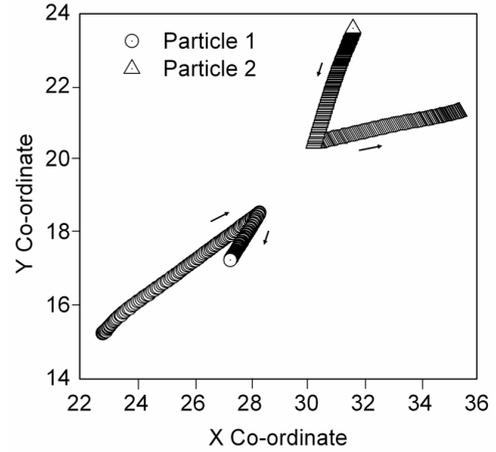


**Fig. 5: Temporal variation of velocity magnitude and direction of two particles where moving Particle 2 hits the stationary Particle 1 from back (very sparse case). There is negligible direction change even after collision for this particular pair (inset).**

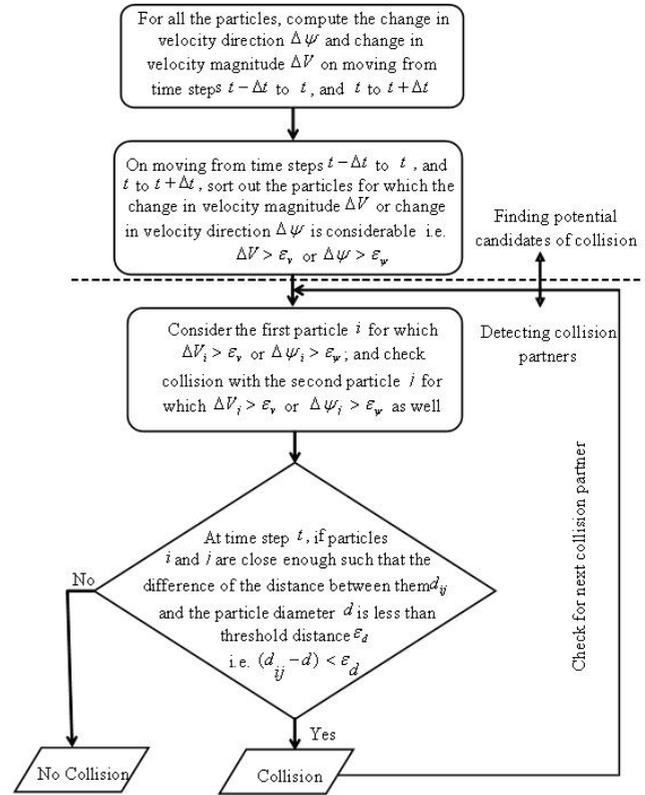
duration-finding algorithm. The criteria concerning changes in velocity vectors sorts out potential candidates for collision, whereas the criteria concerning distance between particle centers confirms the collision partners, as can be seen in Fig. 7. Once contact is detected, the relative velocity of collision partners in the next time step is considered to check whether contact is lost or continued (Fig. 8).

#### Identification of potential candidate of collision

To find potential candidates for collision, the velocity vectors are first calculated using the center coordinates of particles obtained from the PTV software. The velocity of every particle in the streaming and the transverse direction is first calculated using the instantaneous position difference of the center co-ordinates of the particles in between time step  $\Delta t$  i.e.  $V_x = (X(t+\Delta t) - X(t)) / \Delta t$  and  $V_y = (Y(t+\Delta t) - Y(t)) / \Delta t$  from which the change in velocity magnitude  $\Delta V$  in between two time steps is determined. In addition the change in the direction of velocity  $\Delta \Psi$  for each particle is determined using the dot product of velocity vectors of that particle in two successive time steps i.e.  $\Delta \Psi = \cos^{-1}((V(t+\Delta t) \cdot V(t)) / (|V(t+\Delta t)| |V(t)|))$  where  $\hat{V} = \vec{V} / |V|$  is the unit vector. Those particles are considered potential candidates for collision if their velocity magnitude change is greater than a threshold velocity, or if their direction change is greater than a threshold angle on moving from the current time step to the next time step, compared to the values from the previous time step to the current time step. The same methodology is applied in each time step. The magnitude of  $\Delta V$  and  $\Delta \Psi$  are different but optimized for the sparse and dense cases based on observations from several test experiments. This is because in the dense case, the magnitude of the velocity vector change after particle collision is highly diminished due to the limited space available for particles to travel. While defining the magnitude of  $\Delta V$  and  $\Delta \Psi$ , efforts are made to make sure that no potential partners are left unnoticed.



**Fig. 6: Locus of two particles where both the particles are moving towards each other (head on collision), arrow shows the movement direction.**



**Fig. 7: Flow chart for detecting collisions**

#### Detecting collision partners

After detecting potential candidates for collision, the collision partners at time step  $t$  are confirmed by checking the distance  $d_{ij}$  between each of the potential collision candidates at the same time step. If the difference of the distance between the potential collision candidate and the

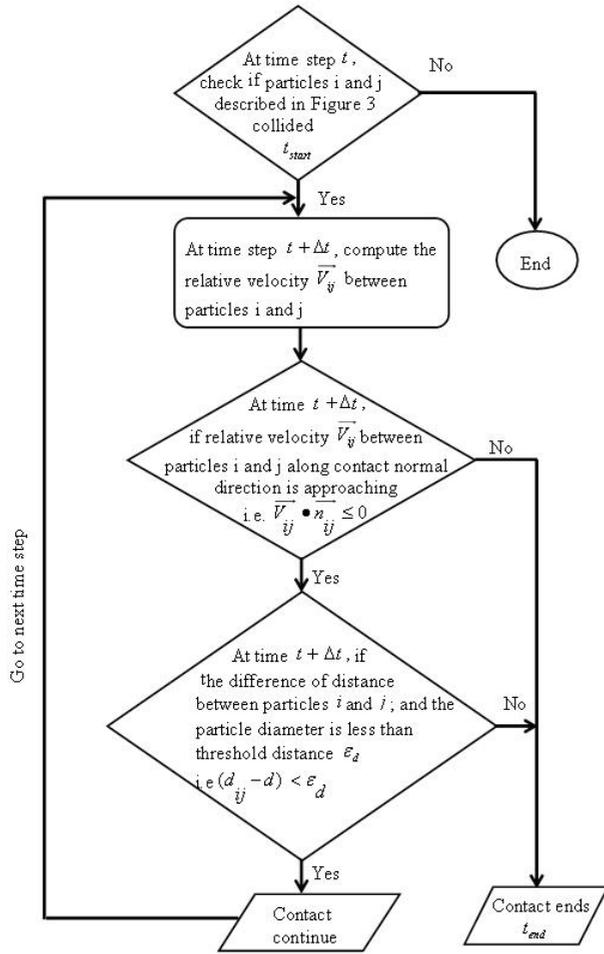


Fig. 8: Flow chart for finding continuing and lost contacts

particle diameter is less than some threshold distance, they are considered collision partners. The magnitude of  $\epsilon_d$  is optimized and set different for sparse and dense cases based on observations from several test experiments and on considering the accuracy of position data.

**ALGORITHM TO FIND OF THE CONTACT IS LOST OR CONTINUED**

To know the status of collision partners in the next time frame, the relative velocity along their contact normal directions, defined by the dot product of the relative velocity vector  $\vec{v}_{ij}$  and the unit normal vector  $n_{ij}$ , on moving from time step  $t + \Delta t$  to  $t + 2\Delta t$  is checked. If it has positive value i.e. going away from each other (separating), the contact is assumed to be lost in time step. However, if it is negative or equal to zero i.e. if the particles are coming closer or move together, the distance between particle centers is checked in time step. If this is within the limit of the threshold distance defined before, that pair is considered to be still in contact; otherwise, it is considered to have lost contact at time step. This process continues until the pair separates. The same

methodology is repeated for all pairs in the shear flow at each time step, and an array of collision status listing collision partners and collision time steps is recorded as output.

**ALGORITHM TO FIND CONTACT DURATIONS**

The time record of the first collision and its separation is kept for all the collision pairs using the collision-detection algorithm and shown graphically in Figs. 7 and 8. The difference between the separation time ( $t_{end}$ ) and the time of first collision ( $t_{start}$ ) gives the contact duration  $t_{ci}$  of that particular pair ('Eq. 1'). If the pairs are collided at time step  $t$  but are separated at time step  $t + \Delta t$ , the contact duration of that particular pair is  $\Delta t$ . However, if the particles are found to be in continuous contact at time step  $t + \Delta t$ , they are followed until they separate, and the difference between the time of separation and that of collision is calculated. If, for example, the pairs are separated at time  $t + n\Delta t$ , the contact duration will be  $n\Delta t$ . The average contact duration of all the pairs in all the time frames calculated using 'Eq. (2)' gives the average contact duration  $t_c$  of that particular flow:

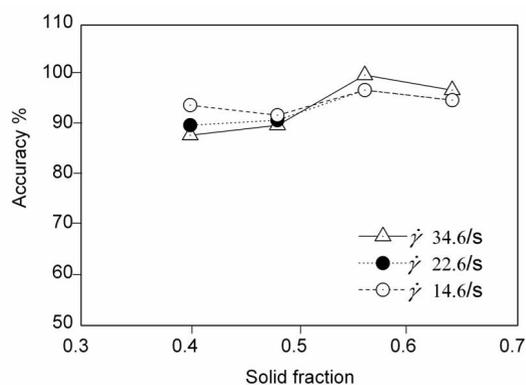
$$t_{ci} = (t_{start} - t_{end}) \dots \dots \dots (1)$$

$$t_c = \frac{1}{N} \times \sum_{i=1}^N t_{ci} \dots \dots \dots (2)$$

where  $N$  is the total number of pairs counted for all the time frames under consideration and  $t_{ci}$  is the contact duration of each pair.

**RESULTS AND DISCUSSIONS**

One of the important questions to answer using the contact detection methodology is whether the proposed method can work effectively and accurately for any combinations of shear rate and solid fractions. This question is genuine because the kinematics of dense and sparse assembly under different deformation rate will be different. Under high shear deformation rate, the change in velocity vector after collision can be very high compared to the slow deformation rate for the same assembly. Considering these differences, we used different threshold value of the velocity vector change to find the potential partners of collision. These values were optimized based on the observations in several test experiments in sparse to dense concentration cases under different deformation rate. Due care is given on the fact that in the sparse assembly under high deformation rate, the particles might make instantaneous collision for very short duration and got separated very soon so that the precision of the high speed video camera might not be sufficient to capture that collision. For this, the threshold distance to confirm the collision is assigned high values for such a flow. In the dense and slow case on the other hand, particles have very little space to travel and hence, the distance travel by particles in each collision will be very small, which can be related also with the mean separation

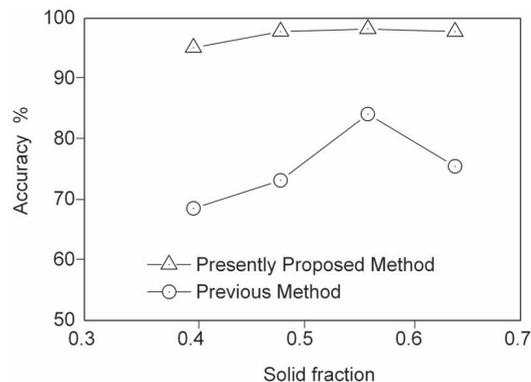


**Fig. 9: Accuracy of proposed method of collision detection as compared to the visual observation of digital video in particle tracking software (lines are drawn only to make it easy to follow the points).**

distance or free flight time. Hence, lower threshold values are used for such a flow.

To find the accuracy of the collision-detection algorithm, the results obtained are compared with those visually observed in the digital video for sparse to dense concentration cases. Comparisons for 40 random observations for sparse to dense concentration cases reveal that on average, the algorithm overestimated collisions by about 5 % compared to visual observations, which can be considered reasonably good estimation in the context of difficulty on getting the particle-particle contacts from the physical experiments. To find the accuracy of the methodology in different shear rate and solid fraction cases, the results for different combinations of shear rate and solid fractions from the algorithm are compared with the observations in the digital video as seen in particle tracking and motion analytical software, DippMotion. The comparison is shown in Fig. 9. In this figure, it can be seen that the average accuracy of the newly formulated contact detection method ranges from almost 90% to 100 %. The sparse concentration cases are found relatively less accurate than the dense concentration flow, which might be partially because the former are more unstable and the fluctuations in velocity is quite high compared to the later. Therefore, the thresholds assigned to find the potential partners might underestimate some of the collisions. However, Fig. 9 also depicts that the accuracy level does not vary seriously with the shear rate.

Moreover, the methodology adopted in Blair and Kudrolli (2003) is applied and the results are compared with the contact detection method formulated in this study. The graph showing the accuracy percentage based on the observations on motion analytical software for both of the methods is plotted in Fig. 10. It can be seen that present methodology has remarkably high accuracy level compared to the previous method. Using presently formulated method the accuracy of the collision detection could be increased by almost 30 % in an average.



**Fig. 10: Comparisons of accuracy of proposed method of collision detection with the previous method where only the velocity direction change criteria were used (lines are drawn only to make it easy to follow the points).**

## CONCLUSIONS

Ability of the experimental setup to inspect pre and post collision phenomena in the physical experiments of granular shear flow provided opportunity to refine the existing contact detection methodology. Newly developed contact detection method is useful also to find the collisions which continue for greater than instantaneous binary collision time. Therefore, it can distinguish even the binary and multi-particle collisions, which is essentially important to study the flow behavior of moderately dense to dense granular flow. Comparisons with the visual observations of the experimental data with the help of particle tracking and motion analytical software reveal that the proposed method can precisely detect the contacts with remarkable accuracy. The method could detect the contacts in all the combinations of shear deformation rate and particle concentration that have been used in this study. It is important to note that presently formulated contact detection method is considerably more accurate compared to the existing methods in experimental granular shear flow.

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