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*Research Article*

# INTEGRATION OF HYPERGEOMETRIC SUPERTRIGONOMETRIC FUNCTIONS USING GENERALIZED HYPERGEOMETRIC FUNCTIONS AND ITS APPLICATIONS

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## ABSTRACT

This study explores the integration of hypergeometric supertrigonometric functions, with a particular focus on hypergeometric supersine and hypergeometric supercosine functions, by leveraging the properties and relationships of generalized hypergeometric functions. By expressing hypergeometric supertrigonometric functions in terms of hypergeometric functions, we derive integral representations that connect these functions to classical results in generalized hypergeometric theory. The work emphasizes the role of convergence conditions, parameters and highlighting special cases where the results can be expressed in simple form, providing a unified framework for evaluating integrals involving hypergeometric trigonometric functions.

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**Keywords:** Clausen hypergeometric; definite integration; generalized hypergeometric; hypergeometric supertrigonometric

## INTRODUCTION AND PRELIMINARIES

The integration of hypergeometric supertrigonometric functions represents an advanced area of mathematical analysis with significant applications in mathematical physics, engineering, and computational mathematics. Hypergeometric functions serve as a unifying framework for various classes of special functions, including hypergeometric supertrigonometric functions (Yang, 2020; Yang, 2021). This study focuses on expressing and evaluating integrals of hypergeometric supertrigonometric functions using the well-established theory of generalized hypergeometric functions in Rainville (Rainville, 1960) . The approach enables a systematic derivation of integral representations, offering insights into their convergence and broader applicability (Binet, 1839; Euler, 1772) .

Hypergeometric functions trace their origins to the work of Euler, Gauss, and Riemann in the 18th and 19th centuries (Bailey, 1935; Euler, 1729; Rainville, 1960). The generalized hypergeometric function, introduced in the 19th century by Clausen. This generalization laid the groundwork for studying a wide variety of special functions. The concept of hypergeometric supertrigonometric functions, arising from extensions of trigonometric and hypergeometric functions. Integration techniques involving hypergeometric functions have been developed, with contributions from mathematicians such as Kummer, Whipple, Saalschütz, paving the way for their modern applications in evaluating complex integrals and solving differential equations (Rainville, 1960; Bailey, 1935). This concept will be applied to double integrals involving generalized hypergeometric supertrigonometric functionst (Basnet *et al.*, 2024a), as well as integrals involving two generalized hypergeometric supertrigonometric functions (Basnet *et al.*, 2024b).

**Definition (Gamma Function):** Gamma function (Euler integral of the first kind) is defined as

$$\Gamma(u) = \int_0^\infty e^{-x} x^{u-1}\, dx, \quad \text{for } u \in \mathbb{C} \text{ with } \mathcal{Re}(u) > 0.$$

This formula was introduced by Euler in 1729 (Euler, 1729),
Where $\Gamma(u+1) = u\Gamma(u)$, $\quad \Gamma(u+1) = u!$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The notation $\Gamma(u)$ was introduced by Legendre in 1814 (Legendre, 1814).

**Definition (Beta Function):** The beta function (Euler integral of the second kind) of $m$ and $n$ is denoted by $B(m, n)$ and is defined by

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 x^{u-1}(1-x)^{v-1}\, dx,\ \mathcal{Re}(u) > 0,$$
$\mathcal{Re}(v) > 0$ for $u, v \in \mathbb{C} \setminus \mathbb{Z}_0^-$,

where $\mathbb{Z}^+, \mathbb{Z}^-, \mathbb{R}^+, \mathbb{R}_- and\ \mathbb{N}$ are the sets of positive integers, negative integers, positive real numbers, negative real numbers and natural numbers, respectively, and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup 0$, $\mathbb{N}_0 = \mathbb{N} \cup 0$.

This formula was first introduced by Euler in 1772 (Euler, 1772). The name of the beta function was introduced by Binet in 1839 (Binet, 1839).

**Definition (Pochhammer symbol):** The Pochhammer symbol is defined as

$(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)}$ and $(u)_n = \prod_{k=1}^{n}(u+k-1), (u)_0 = 1, \ (1)_n = n!$, where $u \in \mathbb{C}$ and $k, n \in \mathbb{N}$.

This symbol was introduced by the German mathematician Leo Pochhammer in 1870 (Pochhammer, 1870) .

**Definition (Gauss's Hypergeometric Function)** (Gauss, 1812)**:** The Gauss's hypergeometric function is defined as

$${}_2F_1 \begin{bmatrix} u, \ v \\ w \end{bmatrix}; \ x \Bigg] = \sum_{n=0}^{\infty} \frac{(u)_n (v)_n}{(w)_n} \frac{x^n}{n!} = 1 + \frac{u\,v}{w}\frac{x}{1!} + \frac{u\,(u+1)v(v+1)}{w\,(w+1)}\frac{x^2}{2!} + \cdots \quad (1.1)$$

where $u, v, w \in \mathbb{C}, n \in \mathbb{N}_0, w \neq 0, -1, -2, \ldots$

This is (i) convergent if $|x| < 1$ and divergent if $|x| > 1$, (ii) convergent if $Re(w-u-v) > 0$ when $x = \pm 1$, and (iii) convergent but not absolutely if $-1 \leq Re(w-u-v) < 0$ when $x = -1$.

**Definition (Kummer Confluent Hypergeometric Function)** (Rainville, 1960)**:** The Kummer confluent hypergeometric function is defined as

$${}_1F_1\begin{bmatrix} u, \\ v \end{bmatrix}; \ x \Bigg] = \sum_{n=0}^{\infty} \frac{(u)_n}{(v)_n}\frac{x^n}{n!} = 1 + \frac{u}{v}\frac{x}{1!} + \frac{u\,(u+1)}{v\,(v+1)}\frac{x^2}{2!} + \cdots \quad (1.2)$$

where $u, v, \in \mathbb{C}, n \in \mathbb{N}_0, v \neq 0, -1, -2, \ldots$
This was introduced by Kummer in 1836.

**Definition (Generalized (Clausen) Hypergeometric Function)** (Bailey, 1935; Rainville, 1960)**:** The generalized (Clausen) hypergeometric function is defined as

$${}_pF_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \sum_{n=0}^{\infty} \frac{(r_1)_n \cdots (r_p)_n}{(s_1)_n \cdots (s_q)_n}\frac{x^n}{n!} \quad (1.3)$$

This is an extended version of the Gauss's hypergeometric function was introduced by Clausen, which is now called the generalized or Clausen hypergeometric function.

This is (i) convergent absolutely if $|x| < 1$ and divergent if $\mathrm{p} < \mathrm{q}$,

(ii) convergent absolutely for $|x| < 1$ if $p = q + 1$ and divergent for $|x| > 1$,

(iii) convergent absolutely for $|x| = 1$ if $Re\left(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k\right) > 0$,

(iv) convergent only for $x = 0$ if $p > q + 1$.

**The Hypergeometric Supertrigonometric Functions via Generalized Hypergeometric function:**

**Definition:** The family of the hypergeometric functions containing the hypergeometric supersine, hypergeometric supercosine, hypergeometric supertangent, hypergeometric supercotangent, hypergeometric supersecant and hypergeometric supercosecant is called the hypergeometric supertrigonometric functions via generalized hypergeometric functions (Yang, 2020, 2021).

**Definition (Hypergeometric Supertrigonometric Functions)** (Yang, 2021)**:**

The hypergeometric supersine function via generalized hypergeometric function is defined as

$${}_p\mathrm{Supersin}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \sum_{n=0}^{\infty} \frac{(-1)^n (r_1)_{2n+1} \cdots (r_p)_{2n+1}}{(s_1)_n \cdots (s_q)_{2n+1}} \frac{x^{2n+1}}{(2n+1)!} \quad (1.4)$$

$${}_p\mathrm{Supercos}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \sum_{n=0}^{\infty} \frac{(-1)^n (r_1)_{2n} \cdots (r_p)_{2n}}{(s_1)_n \cdots (s_q)_{2n}} \frac{x^{2n}}{(2n)!} \quad (1.5)$$

where $a_n, c_n \in \mathbb{C}$ and $n, p, q \in \mathbb{N}_0$.

Both hypergeometric supersine and hypergeometric supercosine via Clausen hypergeometric function are

(i) convergent absolutely for $x \in \mathbb{C}$ if $\mathrm{p} < \mathrm{q}$,

(ii) convergent absolutely for $|x| < 1$ and divergent for $|x| > 1$ if $p = q + 1$ ,

(iii) convergent absolutely for $|x| = 1$ if $Re\left(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k\right) > 0$,

(iv) convergent only for $x = 0$ if $p > q + 1$.

Similarly

$${}_p\mathrm{Supertan}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{\mathrm{pSupersin_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]}{\mathrm{pSupercos_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]},$$

$${}_p\mathrm{Supercot}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{\mathrm{pSupercos_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]}{\mathrm{pSupersin_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]},$$

$${}_p\mathrm{Supersec}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{1}{\mathrm{pSupercos_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]},$$

$${}_p\mathrm{Supercosec}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{1}{\mathrm{pSupersin_q}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Big]}.$$

**Relation between hypergeometric supertrigonometric functions and generalized hypergeometric function** (Yang, 2021)**:**

i. $${}_p\mathrm{Supersin}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{1}{2i}\left\{\mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ ix \Bigg] - \mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ -ix \Bigg]\right\} \quad (1.6)$$

ii. $${}_p\mathrm{Supercos}_q\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ x \Bigg] = \frac{1}{2}\left\{\mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ ix \Bigg] + \mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ -ix \Bigg]\right\} \quad (1.7)$$

**Proof (i):** Let I denote the right-hand side of (i) and using (1.4). Then, we obtain

$$\mathrm{I} = \frac{1}{2i}\left\{\mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ ix \Bigg] - \mathrm{pFq}\begin{bmatrix} r_1, \ldots \ r_p \\ s_1, \ldots, s_q \end{bmatrix}; \ -ix \Bigg]\right\}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(r_1)_n \cdots (r_p)_n}{(s_1)_n \cdots (s_q)_n}\frac{(ix)^n}{(n)!} - \sum_{n=0}^{\infty} \frac{(r_1)_n \cdots (r_p)_n}{(s_1)_n \cdots (s_q)_n}\frac{(-ix)^n}{(n)!}\right]$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(r_1)_n \cdots (r_p)_n}{(s_1)_n \cdots (s_q)_n}\frac{i^n x^n}{(n)!}\left[1 - (-1)^n\right]$$

since $1-(-1)^n = \begin{cases} 2 \ if \ n \ is \ odd \\ 0 \ if \ n \ is \ even \end{cases}$

So that $n$ replaced by $2n+1$, then

$$I = \sum_{n=0}^{\infty} \frac{(r_1)_{2n+1} \cdots (r_p)_{2n+1}}{(s_1)_{2n+1} \cdots (s_q)_{2n+1}} \frac{i^{2n} x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(r_1)_{2n+1} \cdots (r_p)_{2n+1}}{(s_1)_{2n+1} \cdots (s_q)_{2n+1}} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= {}_p\text{Supersin}_q \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right] = \text{LHS}.$$

Similarly, we can prove (ii) also.

**Integral Theorems Involving Generalized Hypergeometric Functions:**

In this section, we present well-established theorems that involve integrals of generalized hypergeometric functions.

**Theorem (2.1):** (Poudel *et al.*, 2023) If $p \le q+1, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0$ and $|x| < 1$. Then

$${}_pF_q \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}F_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ xt \right] dt \tag{2.1}$$

**Corollary (2.1):** (Rainville, 1960) If $Re(r_3) > Re(r_2) > 0$ and $|x| < 1$. Then

$${}_2F_1 \left[ \begin{matrix} r_1, r_2 \\ r_3 ; \end{matrix} \ x \right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3 - r_2)} \int_0^1 t^{r_2 - 1} (1-t)^{r_3 - r_2 - 1} (1-tx)^{-r_1} dt$$

$$= \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3 - r_2)} \int_0^1 t^{r_2 - 1} (1-t)^{r_3 - r_2 - 1} {}_1F_0 \left[ \begin{matrix} r_1 \\ - ; \end{matrix} \ xt \right] dt \tag{2.2}$$

**Corollary (2.2**) (Bateman, 1953) **2)** If $Re(r_2) > Re(r_1) > 0$ and $|x| < 1$. Then

$${}_1F_1 \left[ \begin{matrix} r_1 \\ r_2 ; \end{matrix} \ x \right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{r_2 - r_1 - 1} e^{xt} dt \tag{2.3}$$

**Theorem (2.2):** (Bailey, 1935)

(a) If $Re(\alpha) > 0, \ Re(\beta) > 0, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0, Re(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k) > 0$, $|x| < 1$ and $k \in \mathbb{N}$. Then

$$\int_0^t x^{\alpha - 1} (t-x)^{\beta - 1} {}_pF_q \left[ \begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q ; \end{matrix} \ \lambda x^k \right] dx = B(\alpha, \beta) t^{\alpha + \beta - 1} {}_{p+k}F_{q+} \left[ \begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha + k - 1}{k} \\ s_1, \ldots, s_q, \frac{\alpha + \beta}{k}, \ldots, \frac{\alpha + \beta + k - 1}{k} ; \end{matrix} \ \lambda t^k \right] \tag{2.4}$$

where $\lambda$ is a constant.

(b) If $Re(\alpha) > 0, \ Re(\beta) > 0, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0, p, q \in \mathbb{N}_0, |x| < 1$ and $k, s \in \mathbb{N}$. Then

$$\int_0^t x^{\alpha - 1} (t-x)^{\beta - 1} {}_pF_q \left[ \begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q ; \end{matrix} \ \lambda x^k (t-x)^s \right] dx = B(\alpha, \beta) t^{\alpha + \beta - 1} {}_{p+k}F_{q+k} \left[ \begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha + k - 1}{k}, \frac{\beta}{s}, \ldots, \frac{\beta + s - 1}{s} \\ s_1, \ldots, s_q, \frac{\alpha + \beta}{k+s}, \ldots, \frac{\alpha + \beta + k + s - 1}{k+s} ; \end{matrix} \ \frac{k^k s^s \lambda t^{k+s}}{k+s} \right] \tag{2.5}$$

where $\lambda$ is a constant.

**Theorem (2.3):** (Manocha & Srivastava, 1884) If $Re(\alpha) > 0, \ Re(\beta) > 0, Re(r_1) > 0, \ldots, Re(r_p) \ 0, Re(s_1) > 0, \ldots, Re(s_q) > 0$ and $Re(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k) > 0$. Then

$${}_pF_q \left[ \begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right] dx = B(\alpha, \beta) \ {}_{p+k}F_{q+k} \left[ \begin{matrix} r_1, \ldots, r_p, \ \alpha \\ s_1, \ldots, s_q, \ \alpha + \beta ; \end{matrix} \ 1 \right] \tag{2.6}$$

## RESULTS

In this section, we will evaluate integrals involving hypergeometric Supertrigonometric functions via generalized hypergeometric functions, as stated in the following theorem:

**Theorem:**

If $p \le q+1, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0, p, q \in \mathbb{N}$ and $|x| < 1$. Then

i. $${}_p\text{Supersin}_q \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}\text{Supersin}_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ xt \right] dt \tag{3.1}$$

ii. $${}_p\text{Supercos}_q \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}\text{Supercos}_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ xt \right] dt \tag{3.2}$$

**Proof (i):**

Let I denote the left-hand side of (i) and using (1.4), (1.6) and (2.1). Then

$$I = {}_p\text{Supersin}_q \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ x \right]$$

$$= \frac{1}{2i} \left\{ \text{pFq} \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ ix \right] - \text{pFq} \left[ \begin{matrix} r_1, \ldots r_p \\ s_1, \ldots, s_q ; \end{matrix} \ -ix \right] \right\}$$

$$= \frac{1}{2i} \left\{ \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}F_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ ixt \right] dt - \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}F_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ -ixt \right] dt \right\}$$

$$= \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} \cdot \frac{1}{2i} \left\{ {}_{p-1}F_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ ixt \right] - {}_{p-1}F_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ -ixt \right] \right\} dt$$

$$= \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1 - 1} (1-t)^{s_1 - r_1 - 1} {}_{p-1}\text{Supersin}_{q-1} \left[ \begin{matrix} r_2, \ldots, r_p \\ s_2, \ldots, s_q ; \end{matrix} \ xt \right] dt = \text{R.H.S.}$$

Similarly, we can prove (ii) also.

**Corollary 3.1:**
If $Re(r_3) > Re(r_2) > 0$ and $|x| < 1$. Then

i. $_2\text{Supersin}_1\left[\begin{matrix} r_1, r_2 \\ r_3 \end{matrix}; x\right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3-r_2)} \int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1}\, {_1\text{Supersin}_0}\left[\begin{matrix} r_1 \\ - \end{matrix}; xt\right] dt$ (3.3)

ii. $_2\text{Supercos}_1\left[\begin{matrix} r_1, r_2 \\ r_3 \end{matrix}; x\right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3-r_2)} \int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1}\, {_1\text{Supercos}_0}\left[\begin{matrix} r_1 \\ - \end{matrix}; xt\right] dt$ (3.4)

**Corollary 3.2:**
If $r_1, r_2, x \in \mathbb{C}$ and $Re(r_2) > Re(r_1) > 0$ . Then

i. $_1\text{Supersin}_1\left[\begin{matrix} r_1 \\ r_2 \end{matrix}; x\right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2-r_1)} \int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1} \sin xt\, dt$ (3.5)

ii. $_1\text{Supercos}_1\left[\begin{matrix} r_1 \\ r_2 \end{matrix}; x\right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2-r_1)} \int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1} \cos xt\, dt$ (3.6)

**Theorem (3.2):**
If $Re(\alpha) > 0$, $Re(\beta) > 0, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0, Re\left(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k\right) > 0$ , $|x| < 1$ and $k \in \mathbb{N}$. Then

i. $$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supersin}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k\right] dt = B(\alpha,\beta)\, x^{\alpha+\beta-1}\, {_{p+k}\text{Supersin}_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k}, \ldots, \frac{\alpha+\beta+k-1}{k} \end{matrix}; \lambda x^k\right] \quad (3.7)$$

ii. $$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supercos}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k\right] dt = B(\alpha,\beta)\, x^{\alpha+\beta-1}\, {_{p+k}\text{Supercos}_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k}, \ldots, \frac{\alpha+\beta+k-1}{k} \end{matrix}; \lambda x^k\right] \quad (3.8)$$

where $\lambda$ is a constant.

**Proof (i):**
Let I denote the left-hand side of (i) and using (1.4), (1.6), and (2.5). Then

$$I = \int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supersin}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k\right] dt$$

$$= \frac{1}{2i}\left\{\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; i\lambda t^k\right] - \int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; -i\lambda t^k\right]\right\} dt$$

$$= \frac{1}{2i}\left\{B(\alpha,\beta)\, x^{\alpha+\beta-1}\, {_{p+k}F_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k}, \ldots, \frac{\alpha+\beta+k-1}{k} \end{matrix}; i\lambda x^k\right] - B(\alpha,\beta)\, x^{\alpha+\beta-1}\, {_{p+k}F_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k}, \ldots, \frac{\alpha+\beta+k-1}{k} \end{matrix}; -i\lambda x^k\right]\right\}$$

$$= B(\alpha,\beta)\, x^{\alpha+\beta-1}\, {_{p+k}\text{Supersin}_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k}, \ldots, \frac{\alpha+\beta+k-1}{k} \end{matrix}; \lambda x^k\right] = \text{R.H.S}$$

Similarly, we can prove (ii) also.

**Theorem 3.3:**
If $Re(\alpha) > 0$, $Re(\beta) > 0, Re(r_1) > 0, \ldots, Re(r_p) > 0, Re(s_1) > 0, \ldots, Re(s_q) > 0, p, q \in \mathbb{N}_0$, $|x| < 1$ and $k, s \in \mathbb{N}$. Then

i. $$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supersin}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k (x-t)^s\right] dx = B(\alpha,\beta) x^{\alpha+\beta-1}\, {_{p+k}\text{Supersin}_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k}, \frac{\beta}{s}, \ldots, \frac{\beta+s-1}{s} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k+s}, \ldots, \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}; \frac{k^k s^s \lambda x^{k+s}}{k+s}\right] \quad (3.9)$$

ii. $$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supercos}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k (x-t)^s\right] dx = B(\alpha,\beta) x^{\alpha+\beta-1}\, {_{p+k}\text{Supercos}_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k}, \frac{\beta}{s}, \ldots, \frac{\beta+s-1}{s} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k+s}, \ldots, \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}; \frac{k^k s^s \lambda x^{k+s}}{k+s}\right] \quad (3.10)$$

**Proof (i):**
Let I denote the left-hand side of (i) and using (1.4), (1.6), and (2.5). Then

$$I = \int_0^x t^{\alpha-1}(x-t)^{\beta-1}\, {_p\text{Supersin}_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; \lambda t^k (x-t)^s\right] dx$$

$$= \int_0^x t^{\alpha-1}(x-t)^{\beta-1} \frac{1}{2i}\left\{{_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; i\lambda t^k (x-t)^s\right] - {_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; -i\lambda t^k (x-t)^s\right]\right\} dx$$

$$= \frac{1}{2i}\left\{\int_0^x t^{\alpha-1}(x-t)^{\beta-1} \frac{1}{2i}\{{_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; i\lambda t^k (x-t)^s\right] dx - \int_0^x t^{\alpha-1}(x-t)^{\beta-1} \frac{1}{2i}\{{_pF_q}\left[\begin{matrix} r_1, \ldots, r_p \\ s_1, \ldots, s_q \end{matrix}; -i\lambda t^k (x-t)^s\right] dx\right\}$$

$$= B(\alpha,\beta) x^{\alpha+\beta-1} \frac{1}{2i}\left\{{_{p+k}F_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k}, \frac{\beta}{s}, \ldots, \frac{\beta+s-1}{s} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k+s}, \ldots, \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}; i\frac{k^k s^s \lambda x^{k+s}}{k+s}\right] - B(\alpha,\beta) x^{\alpha+\beta-1}\, {_{p+k}F_{q+k}}\left[\begin{matrix} r_1, \ldots, r_p, \frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k}, \frac{\beta}{s}, \ldots, \frac{\beta+s-1}{s} \\ s_1, \ldots, s_q, \frac{\alpha+\beta}{k+s}, \ldots, \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}; -i\frac{k^k s^s \lambda x^{k+s}}{k+s}\right]\right\}$$

$= B(\alpha,\beta)x^{\alpha+\beta-1}$ ${}_{p+k}$Supersin${}_{q+k}$ $\left[\begin{matrix} r_1,\dots,r_p, \frac{\alpha}{k},\dots,\frac{\alpha+k-1}{k},\frac{\beta}{s},\dots,\frac{\beta+s-1}{s} \\ s_1,\dots,s_q, \frac{\alpha+\beta}{k+s},\dots,\frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}; \frac{k^k s^s \lambda x^{k+s}}{k+s}\right]$ = RHS.

Similarly, we can prove (ii) also.

**Theorem 3.4:**

If $Re(\alpha) > 0,\ Re(\beta) > 0, Re(r_1) > 0,\dots, Re(r_p) > 0, Re(s_1) > 0,\dots, Re(s_q) > 0$ and $Re\left(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k\right) > 0$. Then

i. $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$ ${}_p$Supersin${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; x\right] dx =$ $B(\alpha,\beta)$ ${}_{p+1}$Supersin${}_{q+1}$ $\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; 1\right]$ (3.11)

ii. $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$ ${}_p$Supersin${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; x\right] dx =$ $B(\alpha,\beta)$ ${}_{p+1}$Supersin${}_{q+1}$ $\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; 1\right]$ (3.12)

Proof (i): Let I denote the left-hand side of (i) and using (1.4), (1.6) and (2.6). Then

I= $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$ ${}_p$Supersin${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; x\right] dx$

$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{2i}$ {${}_p$F${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; ix\right] -$ ${}_p$F${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; -ix\right]$}dx

$=\frac{1}{2i}$ { $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{2i}$ {${}_p$F${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; ix\right]$ dx − $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$ ${}_p$F${}_q$ $\left[\begin{matrix} r_1,\dots,r_p \\ s_1,\dots,s_q \end{matrix}; ix\right]$ dx}

$= \frac{1}{2i}$ { $B(\alpha,\beta)$ ${}_{p+1}$F${}_{q+1}$ $\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; i\right] -$ $B(\alpha,\beta)$${}_{p+1}$F${}_{q+1}$$\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; i\right]$ }

$= B(\alpha,\beta)\ \frac{1}{2i}$ { ${}_{p+1}$F${}_{q+1}$ $\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; i\right] -$ ${}_{p+1}$F${}_{q+1}$$\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; i\right]$ }

$= B(\alpha,\beta)$ ${}_{p+1}$Supersin${}_{q+1}$ $\left[\begin{matrix} r_1,\dots,r_p, \alpha \\ s_1,\dots,s_q, \alpha+\beta \end{matrix}; 1\right]$ = RHS

Similarly, we can prove (ii) also.

## SPECIAL CASES

In the main integral theorems, substituting specific numerical values for the parameters results in several significant integrals. These integrals are widely applicable across various practical fields

i. Puttin $p = 3, q = 2, r_1 = r_2 = r_3 = 1$ *and* $s_1 = s_2 = 2$ in (3.1) and (3.2) then we obtain

(a) ${}_3$Supersin${}_2$ $\left[\begin{matrix} 1,1,1, \\ 2,2 \end{matrix}; x\right] = \frac{x}{2^2} - \frac{x^3}{4^2} + \frac{x^5}{6^2} - \frac{x^7}{8^2} + \cdots$ (4.1)

(b) ${}_3$Supercos${}_2$ $\left[\begin{matrix} 1,1,1, \\ 2,2 \end{matrix}; x\right] = 1 - \frac{x^2}{3^2} + \frac{x^4}{5^2} - \frac{x^6}{7^2} + \frac{x^7}{9^2} - \cdots$ (4.2)

ii. Putting $\alpha = 1, \beta = 1,\ p = 1,\ q = 0,\ \lambda = 1$ *and* $r_1 = 1$ in (3.7) and (3.8) then we obtain

(a) $\int_0^x 1\text{Supersin}_0 \left[\begin{matrix} 1 \\ \_ \end{matrix}; t\right] \text{dt} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \cdots$ (4.3)

(b) $\int_0^x 1\text{Supercos}_0 \left[\begin{matrix} 1 \\ \_ \end{matrix}; t\right] \text{dt} =$ $tan^{-1}x,\ where\ |x| < 1.$ (4.4)

iii. Putting $\alpha = 1, \beta = 1,\ p = 1,\ q = 0, and\ r_1 = 1$ in (3.11) and (3.12) then we obtain

(a) $\int_0^1 1\text{Supersin}_0 \left[\begin{matrix} 1 \\ \_ \end{matrix}; t\right] \text{dt} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$ (4.5)

(b) $\int_0^1 1\text{Supercos}_0 \left[\begin{matrix} 1 \\ \_ \end{matrix}; t\right] \text{dt} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (4.6)

## CONCLUSIONS

The integration of hypergeometric supertrigonometric functions using generalized hypergeometric functions provides a unified framework for simplifying complex integrals and uncovering connections to classical results. In this work, we have proved several important theorems involving integrals of hypergeometric supertrigonometric functions, analogous to the theorems associated with generalized hypergeometric functions and highlighted some special cases which expressed in simple form. These findings have broad implications in mathematical Physics, engineering, and computational mathematics, laying a foundation for further research in this area, including summation theorems such as those by Gauss, Kummer, Whipple, Dixon, Saalschutz, and Watson, expressed in terms of gamma and beta functions.

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## AUTHORS CONTRIBUTIONS

GBB: Formulated the research problem, conducted the mathematical analysis, and derived the integration results; NPP: contributed to the literature review, methodology, and verification of results; RPP: assisted with computational analysis, explored applications, and helped in manuscript preparation. All authors collaboratively reviewed and approved the final version of the manuscript.

## CONFLICT OF INTEREST

The authors declare no conflict of interests.

## DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available from the corresponding author, upon reasonable request.

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