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Review Article

A REVIEW ON BURGERS' EQUATIONS AND IT'S APPLICATIONS

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ABSTRACT

This article presents a brief study on the review of the Burgers' equation. It also gives some concepts/ideas and techniques to solve Burgers' equation. Applying Burgers' equation to traffic flow requires concentrated effort for the solution. We develop our insights on how to obtain the Navier-Stokes equation through our inquiry into Burgers' equation. We also demonstrate how the Cole-Hopf transformation for the viscous Burgers' equation is derived. Finally, we use Burger's equation function as a model for the flow of traffic. Additionally, by employing the linear system method, we are able to obtain the answer to the one-way traffic flow problem. The Navier-Stokes equation has been derived to get in-viscid Burger's equation. The principle of Traffic flow and Navier-Stokes models have also been derived.

Keywords: Burgers' equation, fluids dynamics, in-viscid, traffic flow model

INTRODUCTION

Numerous applications of Burgers' equations include gas dynamics, nonlinear elasticity, shallow water theory, geometric optics, combustion theory, cancer treatment, petroleum engineering, irrigation systems, traffic, and mob panic. It frequently manifests as a simplification of more intricate or complex models. Lagerstrom (1949) - Cole-provided an initial illustration. The Burgers equation is produced as a limiting simplified form of the compressible Navier-Stokes momentum equation in the context of the study of viscous compressible fluids in supersonic regime. Burgers' equations are most important because they serve as a foundation for understanding more generic models and how to analyze the behavior of events where the effects of nonlinear transport and dissipation (like viscosity) are inherently incongruent with time.

An important area of mathematics that serves as a model for explaining phenomena that appear in all fields is the study of the Burger equation.

A non-linear partial differential equation of the sort represented by Burgers' equation can have a solution that can be calculated from a linear partial differential equation. It is a fairly straightforward one-dimensional representation of the Navier-Stokes equation. Burger's equation was stated by Bateman as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \quad a< x < b, t < 0 \qquad (1)$$

with initial condition

$$U(x,0) = \emptyset(x), a < x < b \qquad (2)$$

and

$$U(a, t) = f(t) \text{ and } u(b, t) = g(t), t > 0, \qquad (3)$$

where $v > 0$ is the coefficient of kinematic viscosity and $\phi, f \text{ and } g$ are the prescribed function of the variables (Bateman, 1915).

Equation (1) is the steady solution, which was firstly introduced by Bateman. The first term $\frac{\partial u}{\partial t}$ signifies time evolution, and second convection term $u\frac{\partial u}{\partial x}$ denotes non-linearity of shock wave, and third term associated to the viscosity $v\frac{\partial^2 u}{\partial x^2}$ is diffusion term.

It was further treated by Burger in 1948 who gave a model for shock and turbulence and to whom the equation has been named.

LITERATURE REVIEW

The study of the Burgers equation has received a lot of recent research attention. The method most frequently employed is the Cole-Hopf transformation.Burgers' equation is solved in terms of parabolic cylinder functions or Airy functions using the symmetry reduction approach. Recent research by (Veksler and Zarmi, 2007) used exponential wave solutions of the Lax pair connected to the Burgers equation to generate fronts. A helpful study was presented to deal with the perturbed and unperturbed Burgers' equation. A variety of characteristics of freedom and the Normal Form analysis of the perturbed equation were looked into in (Veksler and Zarmi, 2007).

Burgers' equation has been studied for more than 60 years as a straightforward representation of numerous physically intriguing issues and convective diffusion phenomena, including shock waves, turbulence, collapsing free turbulence, traffic flows, and flow-related issues. In a 1915 work by Bateman, the quasi-linear parabolic equation first emerged. This study provides two types of stationary solutions to the infinite domain issue by modeling the motion of a viscous fluid using equations as its viscosity approaches zero. In an effort to create a straightforward mathematical model that illustrates the fundamental traits of turbulence in hydrodynamic flow more than 30 years later, Johannes Martinus Bergers introduced the equation (Burgers, 1948).

There was a ton of research, including equation-related study, after the Hamburger initiative. The equation was officially obtained by Lighthill and Cole (1951) as a quadratic approximation of the unstable Navier-Stokes equation in one dimension. By using the fitted asymptotic expansion method, Fletcher (1982) attained comparable outcomes and observations. using cabbage Independently, Hopf (1950) and Lighthill (1956) discovered that by altering the variables, it is possible to convert the Burgers equation with beginning conditions in the infinite zone to a linear heat equation.

$$\frac{\partial u}{\partial t}(t,x) + u(t,x)\frac{\partial u}{\partial t}(t,x) = \varepsilon\frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x) \qquad (4)$$

The time development of the u(t, x) under nonlinear convection and linear dissipation is described by the Burgers equation, a quasi-linear parabolic PDE. A shock may result from the production of the function $u(t, x)$ when the viscosity is zero. A strong gradient will develop when perturbation ε is small and will be resolved as $t \to \infty$.

In equation (4), the quadratic linear term $\varepsilon\frac{\partial^2 u}{\partial x^2}$ is an elliptic operator.

Numerous characteristics of this issue are particular to non-linearity. Fletcher (1982) provided an explanation for this observation in his dissertation [9]. He emphasized how the Burgers equation serves as a model for the equilibrium between the nonlinear convection term and the diffusion term, which can lead to computational issues in, for instance, fluid mechanics. To elaborate further, if the convection term $\mathbf{uu_x}$ is omitted, equation (4) becomes a classical heat equation. If the diffusion term εu_{xx} is omitted from equation (4), the inviscid Burgers' equation is obtained.

The inviscid Burgers equation is produced if the diffusion term uu_x in equation (4) is left out. Modeling the perturbation convection of an inviscid flow using a hyperbolic equation is possible. A point on the solution of the non-viscous Burgers equation with a large u convect faster than a point with a small u, causing the u to become discontinuous (that is, an impact state) later. Removing the time derivative term from equation (4) leaves the equilibrium equation of Burgers' equation. This is a nonlinear elliptic equation that shows the balance between the convection and diffusion terms. Figure 1 shows a typical solution of an elliptic equation in infinite space. As ε approaches zero, the solution changes discontinuously.

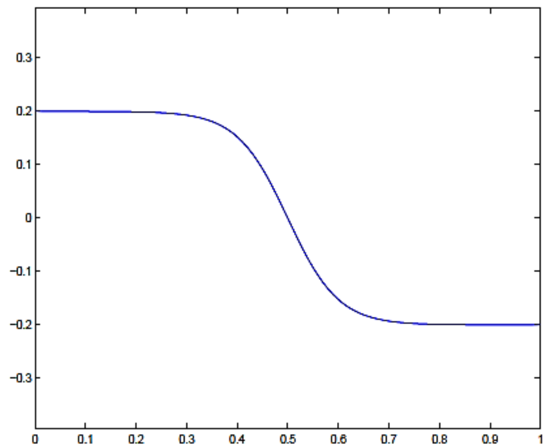

Figure 1. A typical solution of $uu_x = \varepsilon u_{xx}$

Fletcher(1982) described that Burgers' equations can be used to model a variety of issues. Because of the shape of the nonlinear convection factor and the presence of a viscosity term, Burgers' equation can also be seen as a simplified and modified version of the Navier-Stokes equation. Burgers' equation can be converted to the linear diffusion equation and the boundary condition solved. The equation was utilized by Lighthill (1956) as a second order approximation of the unstable one-dimensional Navier-Stokes equations. For a large variety of beginning and boundary conditions, it has proven possible to get the exact solutions to the Burgers equation. Due to the slow convergence of solutions, numerical solutions to the Burgers equation were shown to be impractical for tiny viscosities. The viscid solution demonstrated how viscid form is derived using the Cole-Hopf transformation. To arrive at the viscous Burgers equation, they solved the Navier-Stokes equation. Additionally, they included a traffic flow in viscid Burgers' equation.

In macroscopic car-following models, Nagatani (2002) used analytical and numerical methods to study the phenomenon of traffic bottlenecks. Different density waves can be seen in the traffic flow from uncongested, lower density traffic to crowded, higher density traffic. The non-linear map is used to explain the bunching and delay of the vehicles.

Despite the fact that there is a vast amount of material on Burgers' equation. Through the equation's maximal sub algebras, Ouhadan *et al.* (2011) were able to obtain some exact solutions, including exponential, rational, and periodic ones that formed a new class of Lie point symmetries. The modified Hopf-Cole transformation is used to create the precise solutions to the forced Burgers' equation for both stationary and transient external forces.

DISCUSSION AND METHODS

The Navier- Stokes equation and in viscid Burgers equation has been applied in this paper. The Burger's equation

$$\frac{\partial y}{\partial t}+u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2} \quad (5)$$

Is a mathematical model of viscous Burger's equation and when v=0 then it takes form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}= 0 \quad (6)$$

is in viscid Burgers' equations.

In (6), u, ν, x and t are respectively the speed, kinematic viscosity, spatial coordinate and time.

Burgers' Model

Burgers' model is a mathematical tool that is used to understand some of the inside behavior of the general problem. (Landajuela, M. 2011) has presented two models which are very useful. Now the two examples are mentioned and discussed below.

Navier-Stokes Burger's equations model

Suppose the Navier -Stokes equations

$$(\rho v)_t+\nabla.(\rho v v)+\nabla p-\mu\nabla^2 v=0 \quad (7)$$

Here $\boldsymbol{\rho}$ is density, p is the pressure, v is velocity and μ be the viscosity of a fluid. But when gravitational effect is negligible, $\boldsymbol{\rho}$t=0 then $\nabla p = 0.$ Equation (7) can also written as

$$\rho\left(\frac{\partial v}{\partial t}+v.\nabla v\right) = \mu\nabla^2 v + F\text{, where F is an external force}$$

And

$$\nabla v = \frac{\partial v}{\partial x}\vec{i}+\frac{\partial v}{\partial y}\vec{j}+\frac{\partial v}{\partial z}\vec{k} \quad (8)$$

Now, substituting the value of ∇v from (8) in equation (7) and breaking the term results in

$$\boldsymbol{\rho}\ \frac{\partial v}{\partial t} + \boldsymbol{\rho} v^x \frac{\partial v}{\partial x} + \boldsymbol{\rho} v^y \frac{\partial v}{\partial y} + \boldsymbol{\rho} v^z \frac{\partial v}{\partial z} + - -\frac{\partial p}{\partial x} - \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) = 0 \quad (9)$$

For no pressure gradient equation (4) reduces to

$$\boldsymbol{\rho}\frac{\partial v}{\partial t}+\boldsymbol{\rho} v^x \frac{\partial v}{\partial x} - \mu\frac{\partial^2 v}{\partial x^2}=0 \quad (10)$$

If viscosity μ=0 then

$$\boldsymbol{\rho}\frac{\partial v}{\partial t}+\boldsymbol{\rho} v^x \frac{\partial v}{\partial x}=0 \quad (11)$$

This equation (11) is used to solve the Burgers' equations.

Traffic flow model

This is an important model that can be used to solve several problems of traffic jam on the road. Traffic flow is of matter of interest to anyone. It includes travelling shocks and refraction waves. The majority of situations that have solutions stem from traffic. Traffic lights used to flash red and green to stop cars in line from colliding.

Suppose the flow of vehicles on the road and let $\boldsymbol{\rho}$(x,t) is the density of vehicles and f(x,t) is the traffic flow. If p^* is prohibited density defined in $0\le p^* \le p_{max}$, the density of vehicles and it's flow must be continuous

$$\frac{\partial p^*}{\partial t^*}+\frac{\partial f}{\partial x^*}=0 \quad (12)$$

The speed of vehicles reduced with increase in density of vehicles on the road, so the flow of vehicles is a function of density gradient that is

$$f= v\, p^* \quad (13)$$

$$fp^*=p^* v(p^*)-c\frac{\partial p^*}{\partial x^*} \quad (14)$$

where, c is constant. When density of vehicles increases on the road or heavy traffic, automatically speed will decrease. We have

$$V(p^*) = \frac{v_{max}}{p_{max}}(p_{max-p^*}) \quad (15)$$

Substituting the equation (14) and equation (15) into equation (13) then

$$\frac{\partial p^*}{\partial t^*}+\frac{d}{dx^*}\left[\frac{v_{max}}{p_{max}}-p^*\right]=c\,\frac{\partial p^{*2}}{dx^{*2}}(x) \quad (16)$$

This equation (16) can be written in another form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[\left(\alpha+\frac{1}{2}\ \beta\rho\right)\right] \quad (17)$$

This is the Burgers' traffic flow model.

$\alpha = v_m$= Maximum Velocity

β = -2 $\frac{v_m}{p_m}$ = Density

CONCLUSIONS

The study concludes the brief review, idea, and concepts about the Burgers' equation and its applications in real life. The concepts developed by various researchers has also been presented in this study. The inviscid Burgers' has been used to derive the burger's equation and applied this equation into the traffic flow. The principles of Navier-stokes model and Traffic flow model have also been discussed.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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