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Research Article

ON THE PARTITION OF FAST ESCAPING SETS OF A TRANSCENDENTAL ENTIRE FUNCTION

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ABSTRACT

For a transcendental entire function *f*, the set of form $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$ is called an escaping set. The major open question in transcendental dynamics is the conjecture of Eremenko, which states that for any transcendental entire function *f*, the escaping set $I(f)$ has no bounded component. This conjecture in a special case has been proved by defining the fast escaping set $A(f)$, which consists of points that move to infinity as fast as possible. Very recent studies in the field of transcendental dynamics have concentrated on the partition of fast escaping sets into maximally and non-maximally fast escaping sets. It is well known that a fast escaping set has no bounded component, but in contrast, there are entire transcendental functions for which each maximally and non-maximally fast escaping set has uncountably many singleton components.

Keywords: Escaping set, fast escaping set, maximally fast escaping set, non-maximally fast escaping set.

AMS (MOS) Subject Classification: 37F10, 30D05

INTRODUCTION

We denote the set of integers greater zero by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, and the complex plane by $\mathbb{C}$. We assume the function $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function (TEF) unless otherwise stated. For any $n \in \mathbb{N}$, f^n always denotes the *nth iteration* of *f*. For any $z \in \mathbb{C}$, the set $\{f^n(z)\}_{n\geq 0}$ of iterates of f is called the *orbit* of z. A family $\mathcal{F} = \{f : f$ is meromorphic on the Riemann sphere$\}$ forms a *normal family* if every sequence $\{f_n\}_{n\in\mathbb{N}}$ of the functions contains a subsequence which converges uniformly to a finite limit or converges to ∞ on every compact subset D of $\mathbb{C}$. The *Fatou set* $F(f)$ of f is the set of points $z \in \mathbb{C}$ such that the sequence $\{f^n(z)\}_{n\geq 0}$ forms a normal family in the neighborhood of z. A connected subset of the Fatou set is called a *Fatou component*. The Julia set is the complement of the Fatou set in the complex plane, and it is denoted by $J(f)$. The basic properties and structure of these sets can be found in the work of Bergweiler (1993), Carleman (1992), Hua and Yang (1998), Milner (2006), Morosawa et al. (1999).

In recent years, much interest and more effort have been devoted to understanding the structure and properties of the escaping set $I(f)$ of f which is defined as follows:

Definition 1. *Let f be a transcendental entire function. Let us define*

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

This is called an escaping set. Any point $z \in I(f)$ is called an escaping point.

For a TEF *f*, the escaping set $I(f)$ was first studied by Eremenko (1989) together with formulation of the following conjecture.

Conjecture 1. *Each component of $I(f)$ is unbounded.*

The conjecture has been proved by using the fast escaping set $A(f)$, which consists of points whose iterates tend to infinity as fast as possible. This set was first introduced by Bergweiler and Hinkkanen (1999) and defined in the following form by Rippon and Stallard (2012).

Definition 2. *Let f be a transcendental entire function. The set of the form*

$$A(f) = \{z \in \mathbb{C} : \exists\, L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R, f) \text{ for } n \in \mathbb{N}\}$$

is called a fast escaping set, *where* $M(r, f) = \max_{|z|=r} |f(z)|$, $r > 0$ *and* $M^n(r, f)$ *denotes iteration of* $M(r, f)$ *with respect to r.* $R > 0$ *is any value such that* $M(r, f) > r$ *for* $r \geq R$.

This paper is the most recent study in the field of transcendental dynamics and is primarily concerned with the partition of the fast escaping set $A(f)$. The partition of $A(f)$ was introduced by Sixsmith (2015) on the basis of the following sets.

Definition 3. *Let f be a TEF. The set of the form*

$$A^+(f) = \{z \in A(f) : \exists\, N \in \mathbb{N} \text{ such that } |f^n(z)| = M(|f^{n-1}(z)|, f) \text{ for } n \geq N\}$$

is called the maximally fast escaping set and its complement in $A(f)$*, that is, the set*

$$A^-(f) = A(f) \,/\, A^+(f)$$

is called the non-maximally fast escaping set.

MATERIALS AND METHODS

In this section, we discuss fast escaping set and its properties and partitions (that is, maximally and non-maximally fast escaping sets).These properties will be useful tools for our further investigations and our desired results.

The fundamental properties of the fast escaping set of a transcendental entire function are as follows.

Theorem 1. *For a TEF f, the following statements are hold.*

1. $A(f) = A(f^n)$ *for* $n \geq 2$.
2. $A(f) \neq \emptyset$.
3. $A(f)$ *is completely invariant.*
4. $A(f)$ *is independent of* R.
5. $J(f) \cap A(f) \neq \emptyset$.
6. $J(f) = \partial A(f)$.
7. $A(f)$ *has no bounded components.*
8. $U \subset A(f)$ *for any Fatou component that meets* $A(f)$.
9. $\overline{U} \subset A(f)$ *for any multiply connected Fatou component* U.
10. $J(f) = \overline{A(f) \cap J(f)}$

Rippon and Stallard (2005) proved statement (1), (5) and (6); Bergweiler and Hinkkanen (1999) proved statements (2), (3), and (4); and Rippon and Stallard (2012) also proved statements (7), (8), (9), and (10) respectively in [Theorems 1.1, 1.2, 4.4, 5.1(c)].The result (7) is an important one and is considered the strongest result in the direction of Eremenko's conjecture.

The following results are due to Sixsmith [Theorems 2, 3] (2015), which are nothing other than the properties of maximally and non-maximally fast escaping sets.

Theorem 2. *Let f be a TEF. Then the following statements are hold.*

1. *$A^+(f)$ and $A^-(f)$ are completely invariant sets.*
2. *$A^-(f) \cap J(f) \neq \emptyset$.*
3. *If U is a simply connected Fatou component of f that meets $A(f)$, then $U \subset A^-(f)$.*
4. *If f is a multiply connected Fatou component of f, then $U \cap A^-(f) \neq \emptyset$.*
5. *$A^-(f)$ is dense in $J(f)$ and $J(f) \subset \partial A^-(f)$.*
6. *If $ $A^+(f) \cap J(f) \neq \emptyset$, then $A^+(f)$ is dense in $J(f)$ and $J(f) \subset \partial A^+(f)$.*
7. *If $A^+(f) \cap F(f) = \emptyset$, then $J(f) = \partial A^-(f)$.*
8. *If $A^+(f) \cap J(f) \neq \emptyset$ and $A^+(f) \cap F(f) = \emptyset$, then $J(f) = \partial A^+(f)$.*

The proof of the statement (1) follows from the Definition.3, the statement (2) is proved by using new covering results for annuli, and it is used to construct point in $A^-(f) \cap J(f)$ in the case of TEF with no multiply connected Fatou component. The rest of the statements are followed by the well-known distortion lemma of Bergweiler [Lemma 7] (1993) and Rippon and Stallard [Lemma 10] (2011).

The statement (7) of Theorem 1 may not hold for the sets $A^+(f)$ and $A^-(f)$, a quite contrasting result due to Sixsmith [Theorem 4] (2015). He reached to the following conclusion by considering the function of Hardy (1909).

Theorem 3. *There is a TEF f such that*

1. *$A^+(f)$ is uncountable and totally disconnected.*
2. *$A^-(f)$ has uncountably many singleton components and at least one unbounded component.*

Actually, Sixsmith [Example 5] (2015) defined a family of transcendental entire functions by

$$F_\alpha(z) = \alpha \exp(e^{z^2} + \sin z) \text{ for } \alpha > 0$$

The function f_1 considered by Hardy (1909). Theorem 3 follows from the function $f_\alpha(z)$ on the basis of the following results.

Theorem 4. *For the function $F_a(z) = a \exp(e^{z^2} + \sin z)$ for $a > 0$, we have*

1. *$A^+(f)$ is uncountable and totally disconnected,*
2. *if $a > 0$ is sufficiently small, then there are uncountably many singleton components of $A^-(f)$ with at least one unbounded component.*

For the proof, we refer to [Lemma 5.1, 5.2] Sixsmith (2015).

RESULTS AND DISCUSSION

In this section, we show our results as further investigations of maximally and non-maximally fast escaping sets of a transcendental entire function.

Further investigation on Maximally Fast Escaping sets

In this subsection, we attempt to identify additional structure and properties of the sets $A^+(f)$. Inspection lead us to believe that the set is at most $A^+(f)$ a small subset of $A(f)$. This type of inspection is really true and such a result is obtained on the basis of the following definition.

Definition 4. *Let f be a TEF. The set of the form*

$$\mathcal{M}(f) = \{z \in \mathbb{C} : |f(z)| = M(|z|, f)\}$$

is called the maximal modulus set, and it consists of points where function f attains maximum modulus.

This set is formally defined by Taylor (2000) but was already used by Valiron (1949). It was proved that the set $\mathcal{M}(f)$ consists of, at most, a countable union of maximal curves, which are analytic except at their endpoints. On the basis of this result, a maximally fast escaping set $A^+(f)$ enjoys the following additional result.

Theorem 5. *Let f be a TEF. Then $A^+(f)$ is contained in a countable union of curves, which are analytic except at their endpoints.*

Proof: If $z \in A^+(f)$, then there exists $N \in \mathbb{N}$ such that $f^n(z) \in \mathcal{M}(f)$ for $n \geq N$, according to Definition 4. Which is equivalent to $z \in f^{-n}(\mathcal{M}(f))$. Thus, $A^+(f) \subset \bigcup_{i=0}^{\infty} f^i(\mathcal{M}(f))$. This shows that $A^+(f)$ is contained in the countable union of curves, each of which is analytic except possibly at its endpoint.

What does it mean to have a maximally fast escaping point? We seek an answer from the following construction by Sixsmith (2012). Define a function

$$R_A(z) = \max\{R \geq 0: M^n(R, f) \to \infty \text{ as } n \to \infty \text{ and } |f^n(z)| \geq M^n(R, f) \text{ for } n \in \mathbb{N}\}$$

Note that $R_A(z) \geq 0$ when right hand side for the expression $R_A(z)$ is non-empty and otherwise it is set by -1.

Theorem 6. *Suppose f is a TEF and* $R_A(z) \geq 0$. *Then for any* $z \in A^+(f)$, *there exists* $N \in \mathbb{N}$ *such that*

$$|f^n(z)| = M^n(R_A(z), f) \text{ for } n \geq N.$$

Proof: Let $z \in A^+(f)$, then by Definition 4, there exists $N \in \mathbb{N}$ such that

$$|f^n(z)| = M(|f^{n-1}(z)|, f) \text{ for } n \geq N$$

Set $R = |f^{N-1}(z)|$, then $M^n(R, f) \to \infty$ as $n \to \infty$ and $|f^n(z)| = M^{n+1-N}(R, f)$ for $n \geq N$. Since $R_A(z) \geq 0$, so that

$$M^{N-1}(0, f) \leq M^{N-1}(R_A, f) \leq |f^{N-1}| = R$$

From this last inequality, there exists $R' \geq R_A(z)$ such that $M^{N-1}(R') = R$. Thus

$$|f^n(z)| = M^n(R', f), \text{ for } n \geq N$$

Since $R_A(z) = \max\{R, R'\}$. Hence, we must $R_A(z) = R'$.□

Next, we consider maximally fast escaping set of some particular example of TEFs. The following examples are cited from the work of Sixsmith (2015).

Example 1. [Example 1, 2 Sixsmith (2015)] Let $f(z) = e^z$. Then $\mathcal{M}(f) = [0, \infty)$. We see $f(\mathcal{M}(f)) \subset \mathcal{M}(f)$. By Theorem 5, $A^+(z)$ is countable union of analytic curves

$$A^+(z) = \bigcup_{i=1}^{\infty} f^{-1}([0, \infty))$$

And for the function $g(z) = ie^z$, we have $\mathcal{M}(f) = [0, \infty)$, but $f(\mathcal{M}(f)) \cap \mathcal{M}(f) = \emptyset$. So $A^+(z) = \emptyset$.

Example 2. [Example 6, Sixsmith, (2015)]Let $g_\lambda(z) = \lambda e^z$ for $0 < \lambda < 1/e$. Then g_λ has an unbounded simply connected Fatou component which contains the imaginary axis and attracting fixed point and also the repelling fixed point $q > 1$. Furthermore, $(q, \infty) \subset A^+(f)$, where (q, ∞) is a component of $A(f)$. This example showed that the unbounded component of the fast escaping set is contained in the maximally fast escaping set.

Further Investigation on Non-maximally Fast Escaping Set

In this subsection, we divide the non-maximally fast escaping set $A^-(f)$ further into two sets, which consist of points with extremely fast and moderately fast escape rates. By Definition 3, the non-maximally fast escaping set looks like

$$A^-(f) = \{z \in A(f): \exists N \in \mathbb{N} \text{ such that } |f^n(z)| > M(|f^{n-1}(z)|, f) \text{ for } n \geq N\}$$
$$\cup \{z \in A(f): \exists N \in \mathbb{N} \text{ such that } |f^n(z)| < M(|f^{n-1}(z)|, f) \text{ for } n \geq N\}$$

It is obvious that the set $A^-(f)$ can be further partitioned into the two sets of the form

$$A'(f) = \{z \in A(f): \exists N \in \mathbb{N} \text{ such that } |f^n(z)| > M(|f^{n-1}(z)|, f) \text{ for } n \geq N\}$$

and

$$A''(f) = \{z \in A(f): \exists N \in \mathbb{N} \text{ such that } |f^n(z)| < M(|f^{n-1}(z)|, f) \text{ for } n \geq N\}$$

We call these sets, respectively, the *extremely fast escaping set* and *moderately fast escaping set*. Note that that the set $A'(f)$ consists of points whose rate of escape is extremely fast and the set $A''(f)$ consists of points whose rate of escape is moderately fast. The fast escaping set $A(f)$ can now be divided into three disjoint subsets; the maximally fast escaping set $A^{-}(f)$, the extremely fast escaping set $A'(f)$, and moderately fast escaping set $A''(f)$.

CONCLUSIONS

Mainly, we concentrated on the partition of the fast escaping set of a transcendental entire function into two subsets, the maximally fast escaping set and the non-maximally fast escaping set. We found that these sets have strong dynamical properties. It was shown by Rippon and Stallard that the fast escaping set has no bounded components. In contrast, we found that the maximally and non-maximally fast escaping sets each have uncountably many singleton components.

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CONFLICT OF INTEREST

The authors declare no competing interest.

DATA AVAILABILITY STATEMENT

All the information's that support the findings of this study are available from the corresponding author.

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