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***Research Article***

# THE LAW OF THE ITERATED LOGARITHM

**Santosh Ghimire[1,*], Hari Thapa[2]**
[1]*Department of Applied Sciences and Chemical Engineering, Pulchowk Campus, Tribhuvan University*
[2]*Kathford College of Engineering and Management, Tribhuvan University*
**Corresponding Author: santoshghimire@ioe.edu.np*
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## ABSTRACT

The article begins first with the history and the development of the law of the iterated logarithm, abbreviated LIL. We then discuss the LIL in the context of independent random variables, dyadic martingales, lacunary trigonometric series, and harmonic functions. Finally, we derive a LIL for a sequence of dyadic martingales.

***Keywords***: Dyadic martingales, independent random variables, lacunary series, law of the iterated logarithm

## INTRODUCTION

The LIL is a celebrated law in the various domains of mathematics. There has been a tremendous amount of work on the LIL in various contexts in mathematics. In the theory of probability, the LIL operates roughly in between the central limit theorem and the law of large numbers. Some of the contexts where the LIL has been established are independent random variables, martingales, harmonic functions, lacunary trigonometric series, Brownian motion, Gaussian process, Bloch functions to name just a few. We first begin with the context where the first LIL was introduced, and this will be followed by the various historical developments on it. In order to do so, we need to recall some definitions and theorems.

**Definition 1.1** The decimal and dyadic expansion of a number N is [0,1) are:

$$\mathrm{N} = \sum_{n=1}^{\infty} \frac{X_n}{10^n};\ X_n \in \{0,1,2,\ldots\ldots,9\}\quad \mathrm{N} = \sum_{n=1}^{\infty} \frac{X_n}{2^n};\ X_n \in \{0,1\}$$

Then N is said to be a normal number with base 10 and 2 respectively if $\lim_{n\to\infty} \frac{\omega_k^n(N)}{n} = \frac{1}{10}$ and $\lim_{n\to\infty} \frac{\omega_k^n(N)}{n} = \frac{1}{2}$ where $\omega_k^n(N)$ is the number of times the digit $k, 0 \le k \le 9$ appear in the expansion of N.
Here is the result of Borel on the normal numbers.

**Theorem 1.2** (Borel). *For a number $t \in [0,1)$ in its expansion, let $N_n(t)$ denote the number of times 1 appear in the first n-places. Then we have $\lim_{n\to\infty} \frac{N_n(t)}{n} = \frac{1}{2}$ for a.e. t in Legesgue measure.*

The above theorem of Borel gives the rate of convergence, and this rate was later on improvised by many mathematicians, namely Hausdorff, Hardy and Littlewood and Khintchine. The exact rate of convergence was given by A. Khintchine (Khintchine, 1924). This result of Khintchine is considered as the earliest LIL in the theory of probability. His result is:

**Theorem 1.3** (Khintchine). *For a number $t \in [0,1)$, let $N_n(t)$ denote the number of times 1 appear in the binary expansion of t in the first n-places, then*

$$\lim_{n\to\infty} sup \frac{|S_n|}{\sqrt{2n \log logn}} = 1$$

The above theorem is the primitive LIL result in the theory of probability and is called Khintchine's LIL. The reason behind the name "law of the iterated logarithm" has been given to this law is due to the iteration log log*n*. Then this LIL result of Khintchine was later generalized by N. Kolmogorov in the bigger context of independent random variables. After the result of Kolmogorov, the LIL got much attraction and the mathematicians started to think about the analogue of this result in various other directions. The areas where a tremendous amount of research has been conducted on the LIL are independent random variables, lacunary trigonometric series, dyadic martingales and harmonic functions. We discuss the result on these contexts in detail. We begin with LIL in the context of independent random variables.

## THE LIL FOR INDEPENDENT RANDOM VARIABLES

N. Kolmogorov (Kolmogorov, 1929) was the one who revolutionized the LIL. To this end, we first record his LIL result for the independent random variables. Slightly abuse of language, we use i.d.r.v for random variables which are independent and identically distributed with mean zero and variance one and we write c.d.f. for common distribution function.

**Theorem 2.1** (Kolmogorov,1929). Let $S_n = \sum_{i=1}^{n} X_i\ where\ X_n$ is i.d.r.v. Assume that $|X_n|^2 \le$

$\frac{\varepsilon_n\, n}{\log\log n}$ for some constants $\epsilon_n \to 0$. Then for a.e. $\omega$, $\lim_{n\to\infty} sup \frac{|S_n|}{\sqrt{2n \log logn}} = 1$

The above result is Kolmogorov's LIL and is considered as principal accomplishment in the theory of probability. In 1941, Hartman-Winter (Hartman & Winter, 1941) also obtained a LIL for identically distributed independent random variables. This result is also popularly known as limsup LIL.

**Theorem 2.2** (Hartman-Winter, 1941). *Let $\{X_k\}_{k\geq 1}$ be a sequence where $X_k$ is i.d.r.v. and $S_n$ denotes the $n^{th}$ partial sums of the random variables, then $\lim_{n\to\infty} sup \frac{S_n}{\sqrt{2n\, log\, logn}} = 1$ almost surely.*

K.L. Chung (Chung, 1948) then introduced the liminf version of the above result. We now state the theorem by Chung.

**Theorem 2.3** (Chung, 1948). *If F is c.d.f. of a sequence i.d.r.v. $\{X_n; n \geq 1\}$ with variance $\sigma^2$, and also assume $E(|X|^3) < \infty$. Then $\lim_{n\to\infty} inf \sqrt{\frac{\log logn}{n}} \max_{1\leq j\leq n} |S_j| = \frac{\sigma\pi}{\sqrt{8}}$ with probability 1.*

We note that the above result was obtained with the extra assumption of finite third moment. Under the same assumption of Hartman-Winter, Xia Chen [29] established a LIL where he has considered the limit instead of limit supremum.

**Theorem 2.4** (Chen, 2015). *Let $\{X_k\}_{k\geq 1}$ denote a sequence of i.d.r.v. and $S_k$ denotes the $k^{th}$ partial sums of the random variables, then $\lim_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \max_{1\leq k\leq n} \frac{S_k}{\sqrt{k}} = 1$ almost surely.*

In connection with the independent random variables, we state another LIL for Ornstein-Uhlenbeck process (Chen, 2015). For this, we first recall the definition:

**Definition 2.5.** Let $\{X(t); t \geq 0\}$, be a stationary, centered and continuous Gaussian process. Then it is called one dimensional Ornstein-Uhlenbeck process if $Cov(X(0), X(t)) = e^{-\frac{t}{2}}, t \geq 0$.

**Theorem 2.6** (Chen, 2015). *Let $\{X(t);\, t \geq 0\}$ be an Ornstein-Uhlenbeck process. Then*

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\, log\, t}} \max_{s\leq t} X(s) = 1 \text{ almost surely.}$$

## THE LIL FOR PARTIAL SUMS OF LACUNARY SERIES

The context of lacunary trigonometric series is considered as the first context in analysis where the LIL was introduced. We remark that the sums of lacunary series exhibits the behaviors which are similar to independent random variables. Due to this reason, they are also called weakly dependent random variables. We first begin with the definition:

**Definition 3.1** (Lacunary Series). A $q-$ lacunary series is a series of the form given by $S_m(\theta) = \sum_{k=1}^{m}(a_k cos\, n_k\theta + b_k sin\, n_k\theta)$ satisfying lacunary condition $\frac{n_{k+1}}{n_k} > q > 1$.

Inspired by the LIL of Kolmogorov, Salem and Zygmund obtained the analogue result for lacunary series. The LIL introduced by Salem and Zygmund is considered as the opening result in the analytic setting (Banelos & Moore, 1991). We next record their theorem:

**Theorem 3.2** (R. Salem and A. Zygmund, 1950). *Let $S_m$ denote a $q-$lacunary series and the $n_k$ be positive integers. Write $B_m^2 = \frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)$ and $M_m = \max_{1\leq k\leq m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$.* If $\lim_{m\to\infty} B_m = \infty$ *and* $S_m$ satisfying $M_m^2 \leq K_m \frac{B_m^2}{loglog(e^e + B_m^2)}$ for $K_m \downarrow 0$.

Then $\lim_{m\to\infty} sup \frac{S_m(\theta)}{\sqrt{2B_m^2\, log\, logB_m}} \leq 1$ $a.e.\, \theta \in T$, unit circle.

Note that $\int_{-\pi}^{\pi} S_m(x)dx = 0$ and $\sigma = B_m = \sqrt{\frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)}$.

We note that in the assumption $n_k$ is a positive integer. In the case of integer, Erdös and Gal (1955) generalized the result of Salem and Zygmund. They considered a special form of the series and attained the LIL as follow:

**Theorem 3.3** (Erdös & Gal, 1955). *Let $n_k$ be integers and $S_m$ be a lacunary series given by $S_m(\theta) = \sum_{k=1}^{m} exp(in_k\theta)$ and are integers. Then $\lim_{m\to\infty} sup \frac{S_m(\theta)}{\sqrt{m\, log\, logm}} = 1$ for a. e. $\theta$ in the unit circle where a. e. stands for almost every.*

In 1955, M. Weiss while doing here Ph.D. was able to obtain the counterpart of Kolmogorov's LIL in the above context in full entirety. This result of M. Weiss (Weiss, 1959) is considered as a remarkable result in this area. She obtained this with other remarkable results in early age. Unfortunately, she died at early age. We next record her result:

**Theorem 3.4** (Weiss, 1959). *Let $S_m(\theta) = \sum_{k=1}^{m}(a_k cos\, n_k\theta + b_k \sin n_k\theta)$ be a $q-$lacunary series and $n_k$ be integers. Then under the same assumption of Kolmogorov, we have*

$$\lim_{m\to\infty} sup \frac{S_m(\theta)}{\sqrt{2B_m^2\, log\, logB_m}} = 1 \text{ for a. e. } \theta \text{ in the unit circle.}$$

In all above results of LIL for the given trigonometric series, they only considered only the partial sums up to the $n^{th}$ term. In contrast to these result, Salem and Zygmund obtained a LIL where they used the remainder after n-term of the lacunary series. Their LIL result (Salem & Zygmund, 1950) for this remainder or tail sums is:

**Theorem 3.5** (Salem & Zygmund, 1950). *Let* $S_N^{\sim}(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ *be given lacunary series where* $c_k^2 = a_k^2 + b_k^2$ *satisfying* $\sum_{k=1}^{\infty} c_k^2 < \infty$. *Set* $B_N^{\sim} = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ *and* $M_N^{\sim} = \max_{k\geq N}|c_k|$. *Assume* $B_1^{\sim} < \infty$ *and that* $M_N^{\sim 2} \leq K_N\left(\frac{B_N^{\sim 2}}{loglog\frac{1}{B_N^{\sim}}}\right)$ *for some sequence of numbers* $K_N \downarrow 0$ *as* $N \to \infty$. *Then* $\lim_{N\to\infty} sup \frac{S_N^{\sim}(\theta)}{\sqrt{2B_N^{\sim 2} loglog\frac{1}{B_N^{\sim}}}} \leq 1$ *for a.e.* $\theta$ *in the circle of radius one.*

We remark that the result depends on the tail sums instead of $n^{th}$ sums and this type of LIL is considered as tail LIL. Moreover, we note that the result obtained above has only the upper bound and no lower bound has been of the result has been given. The other inequality of the lower bound of the tail LIL was obtained by Ghimire and Moore (Ghimire and Moore, 2014) under the similar assumption. Their result is:

**Theorem 3.6** (Ghimire and Moore, 2012). Let $S_m$ denote the sum of series given by $S_m(x) = \sum_{k=1}^{m} a_k \cos(2\pi n_k x)$ satisfying $\frac{n_{k+1}}{n_k} \geq q > 1$ *and* $\sum_{k=1}^{\infty} a_k^2 < \infty$. Suppose $\max_{k\geq N} a_k^2 = o\left(\frac{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}{loglog\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}}}\right)$.

Then for a.e. x, $\lim_{n\to\infty} sup \frac{|\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)|}{\sqrt{2\,\frac{1}{2}\sum_{k=N}^{\infty} a_k^2 \;\; loglog\frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}}}} \geq 1$

## THE LIL FOR DYADIC MARTINGALES

A counterpart of Kolmogorov's LIL in the context of dyadic martingale was obtained by W. Stout. Before we state the main result, we first recall the definition of dyadic martingale:

**Definition 4.1** (Dyadic martingales). Let $\mathfrak{F}_n$ denote the $\sigma -$ algebra generated by the dyadic intervals of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ on [0,1). Then a sequence of integrable functions $\{f_n\}_{n=0}^{\infty}$ is called dyadic martingales if it satisfies: $f_n : [0,1) \to \mathbb{R}$ such that for every n, $f_n$ is $\mathfrak{F}_n -$ measurable and $E(f_{n+1}|\mathfrak{F}_n) = f_n$for all $n \geq 0$. We write $d_k(x) = f_k(x) - f_{k-1}(x)$,

$$S_n^2 f(x) = \sum_{k=1}^{n} d_k^2(x), \;\; S^2 f(x) = \sum_{k=1}^{n} d_k^2(x).$$

Asymptotic behavior of dyadic martingales is controlled by the square function defined above. For the long-term nature of dyadic martingales, we first note the result of Burkholder and Gundy (Burkholder & Gundy, 1970):

$$\{x : Sf(x) < \infty\} =^{a.s} \{x : \lim f_n \; exists\}.$$

One can note that asymptotic behavior of $\{f_n\}$ is nice on the set $\{x : Sf(x) < \infty\}$. But how does it behave on the complement of the above set? Precisely, the function $\{f_n\}$ is unbounded almost everywhere on the set $\{x : Sf(x) = \infty\}$. Nevertheless, we can obtain the size of growth of $|f_n|$ on this complement set. The growth of $|f_n|$ on the set where $Sf(x)$ is infinity was entirely answered by W. Stout (Stout, 1970).

**Theorem 4.2** (Stout, 1970). *Let* $\{f_n\}_{n=0}^{\infty}$ *denote a dyadic martingale sequence on the interva [0,1) then we have,*

$$\lim_{n\to\infty} sup \frac{|f_n(x)|}{S_n f(x)\sqrt{2\log\log S_n f(x)}} \leq 1$$

*a.e. on a set with* $\{f_n\}$ *unbounded.*

Now we discuss another direction where the LIL due to Salem and Zygmund was extended: S. Takahashi (Takahashi, 1963) extended the result of Salem and Zygmund by considering a function $f$ satisfying $f(x+1) = f(x), \int_0^1 f(x)dx = 0$. Moreover, let $n_k$ satisfy gap condition. Suppose that $f \in Lip\alpha, 0 \leq \alpha \leq 1$. Then the LIL result obtained by Takahashi is:

(4.1) $$\lim_{N\to\infty} \sup \frac{\sum_{k=1}^{N} f(n_k t)}{\sqrt{N\log\log N}} \leq \text{C a.e.}$$

This result of Takahashi has been further generalized by Dhompongsa (Dhompongsa, 1986), Takahashi (Takahashi, 1988) himself, and Peter (Peter, 2000). In the new generalization, they have taken wide class of functions and considered the weak lacunary condition in contrast to regular condition.

In 2012, Moore and Zhang generalized the LIL of Takahashi with same gap condition but with larger class of functions $f$. Precisely, they considered Dini continuous function $f$. We first define Dini continuous function:

**Definition 4.3** (Dini Continuous). A Dini continuous function is a function $f$ on $\mathbb{R}^n$ that satisfies:

(4.2)
$$\int_0^1 \frac{\omega(f,\delta)}{\delta} d\delta < \infty$$

Moore and Zhang (Moore & Zhang, 2012) extended the result in the case when the function $f$ is Dini continuous. Assuming $f$ satisfying Dini continuous and $n_k$ satisfies the lacunary condition. They obtained a constant $C(n, q, \int_0^1 \omega(\delta)d\delta)$ and

$$\lim_{m\to\infty} sup \frac{|\sum_{k=1}^{m} a_k f(n_k x + c_k)|}{\sqrt{A_m^2 loglog A_m^2}} \leq C \quad a.e.$$

The above result gives the upper bound result. The other inequality of the result was also obtained by Moore and Zhang (Moore and Zhang, 2014). Under the similar conditions as in the upper bound result and with some extra conditions, they obtained the following LIL result.

$$\lim_{m\to\infty} sup \frac{|\sum_{k=1}^{m} a_k f(n_k x + c_k)|}{\sqrt{A_m^2 loglog A_m^2}} \geq C \quad a.e.$$

We remark that in both the upper and lower bound result, the authors have not assumed the $n_k$ to be integers. Moreover, they have not considered any periodicity on the function$f$. The best possible values of the bounds (C and c is still unknown and are open problem in this direction.

We now discuss the LIL in the context o harmonic functions which are considered as an important feature of harmonic analysis. In this setting R. Banelos, I. Klemes and C. N. Moore (Banelos et al., 1988) obtained the counterpart of Kolmogorov's LIL. To state their result, we recall some notations and definitions:

Upper half space and doubly truncated cone respectively defined and denoted by

$$\mathbb{R}_+^{d+1} = \{(y,s): y \in \mathbb{R}^d, s > 0\} \quad \Gamma_\alpha(x,t) = \{(y,s): |x-y| \leq \alpha s, t \leq s \leq 1\}$$

**Definition 4.4** (Lusin area function). The doubly truncated Lusin area function for a harmonic function $u$ in the upper half space is defined as:

$$A_\alpha(u)(x,t) = \left(\int_{\Gamma_\alpha(x,t)} S^{1-d} |\nabla u(y,s)|^2 dyds\right)^{\frac{1}{2}}$$

The LIL introduced by them in the setting of harmonic function is:

**Theorem 4.5** (Banelos, Klemes and Moore, 1988). *If $u$ is harmonic function in $\mathbb{R}_+^{d+1}$, then for a fixed $0 < \beta < \alpha$ and $0 < \gamma < 1$, we have*

$$\lim_{(y,t)\to(x,0),(y,t)\epsilon\Gamma_\beta(x,0)} \frac{|u(y,t)|}{\sqrt{A_\alpha^2(u)(x,\gamma t)\, ln\, ln(A_\alpha(u)(x,\gamma t))}} \leq C$$

*for $a.e. x \in \{x \in \mathbb{R}^d : A_\alpha(u)(x) = \infty\}$. C is a positive constant such that $C(\alpha, \beta, \gamma, d)$.*

They also estimated the other inequality of the result. We now state their result (Banelos et al., 1988):

**Theorem 4.6** (Baneloset al., 1990). *Let $u$ be a harmonic function in $\mathbb{R}_+^{d+1}$ and assume $\alpha > 0$. For $0 < t < 1$, define $K_\alpha(u)(x,t)$ as*

$$A_\alpha^2(u)(x,t) - A_\alpha^2(u)(x,2t) = K_\alpha(u)(x,t) \frac{A_\alpha^2(u)(x,t)}{log\, log(e^e + A_\alpha^2(u)(x,t))}$$

*and set*

$$K_\alpha(u)(x,t) = 1 + \lim_{t\downarrow 0} sup K_\alpha(u)(x,t).$$

*Then*

$$\limsup_{t\downarrow 0} \frac{u(x,t)}{\sqrt{A_\alpha^2(u)(x,t) loglog(A_\alpha(u)(x,t))}} \geq \frac{C_2}{\sqrt{K_\alpha(u)(x)}}$$

*for $a.e. x \epsilon \{x \epsilon \mathbb{R}^d : A_\alpha(u)(x) = \infty$ and $K_\alpha(u)(x) < \infty\}$. $C_2$ is a positive constant depending only on $\alpha$ and $d$.*

There are various other settings where a LIL has been established. We suggest a paper N.H. Bingham (Bingham, 1986) to know about various other contexts where a LIL has been established. For LIL in Banach space, please refer (Ledoux & Talagrand, 1991)For the various results on the LIL for random vectors and for some open problems in LIL, please refer (Liu & Zhang, 2021). Readers are referred to article (Einmahl, 2016) for a LIL in the setting of random walk and in the setting of linear processes, please refer (Hambly et al., 2003). For LIL in the context of Bloch functions, please refer (Prztycki, 1989) and for Brownian motion, refer Qi and Yan (2018).

## MAIN RESULTS

In this section, we prove a LIL for Rademacher functions considering the remainder after $n^{th}$ term. We first state our result.

**Theorem 5.1** *Let $f(t) = \sum_{k=1}^{\infty} a_k r_k(t)$, $f_n(t) = \sum_{k=1}^{n} a_k r_k(t)$, $S_n'^2 f(t) = \sum_{j=n+1}^{\infty} [f_k(t) - f_{k-1}(t)]^2$ where $\{r_k\}_{k=1}^{\infty}$ is the sequence of Rademacher functions and $\sum_{n=1}^{\infty} a_n^2 < \infty$. For a $\theta > 1$, define*

$$n_k = \min\left(n : \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k}\right)$$

*Then,*

$$\limsup_{k\to\infty} \frac{|f(t) - f_{nk}(t)|}{\sqrt{2S_{nk}'^2 f(t) \log\log \frac{1}{S_{nk}'^2 f(t)}}} \leq 1$$

*for a.e. t.*

Before we start the proof, we first recall the definition of Rademacher functions.

**Definition 5.2** (Rademacher functions). Let $r_k$ be a function defined on $[0,1]$ by $r_k(x) = sgn(\sin 2^k \pi x)$ where sgn denotes the signum function. Then the sequence of the functions given by $\{r_k(x)\}_{k=1}^{\infty}$ is called Rademacher functions.

We can note that the Rademacher function gives the value -1 and 1 alternatively and these functions are independent, identically distributed random variables with zero mean and variance one. Moreover, if we define the weighted sums of these functions taking the real numbers, they behave like dyadic martingales. This property will be used in the proof of our main result.

**Lemma 5.3** (Borel-Cantelli) Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the sets $E_k's$.

Proof. For the proof, see (Royden & Fitzpatrick, 2010).

**Proof of the main result**: Note that $\{f_n\}_{n=1}^{\infty}$ is a sequence of dyadic martingale. Clearly $\sum_{i=1}^{n} a_i r_i = f_n$ is measurable function. Moreover, on each $n^{th}$ generation dyadic subinterval $|Q_n| = \frac{1}{2^n}$, one can show:

$$E(f_{n+1}|\mathfrak{F}_n) = \frac{1}{|Q_n|}\int_{Q_n}\sum_{k=1}^{n} a_k r_k(x)dx + a_{n+1}\int_{Q_n} r_{n+1}(x)dx$$

$$= \frac{1}{|Q_n|}\int_{Q_n}\sum_{k=1}^{n} a_k r_k(x)dx + 0$$

$$= \sum_{k=1}^{n} a_k r_k(x) = f_n.$$

Fix $n$. Define a sequence $\{g_m\}$ as follows:

$$g_m(t) = \begin{cases} 0, & if\ m \le n, \\ f_m(t) - f_n(t), & if\ m > n \end{cases} \tag{5.1}$$

We first show that $\{g_m\}$ is a dyadic martingale. Clearly for every m, $g_m$ is measurable with respect to the sigma algebra $\mathfrak{F}_m$. Let $m > n$. Then using the fact that $f_m$ is constant on the cube $Q_m$ we have,

$$\begin{aligned} E(g_{m+1}\ |\mathfrak{F}_m) &= \frac{1}{|Q_m|}\int_{Q_m}[f_{m+1}(t) - f_n(t)]dt \\ &= \frac{1}{|Q_m|}\int_{Q_m} f_{m+1}(t)dt \\ &\quad - \frac{1}{|Q_m|}\int_{Q_m} f_n(t)dt \\ &= \frac{1}{|Q_m|}\int_{Q_m} f_{m+1}(t)dt - f_n(t) \\ &= f_m(t) - f_n(t) = g_m(t). \end{aligned}$$

Thus, we have $E(g_{m+1}\ |\mathfrak{F}_m) = g_m$. This shows that $\{g_m\}$ is a dyadic martingale. Now we make the used of the following inequality whose proof can be found in (Ghimire, 2020)

**Inequality 5.4.** For a dyadic martingale $\{f_n\}$ and $\lambda > 0$ we have

$$\left|\left\{t \in (0,1]: \sup_{m\ge1}|f_m(t)| > \lambda\right\}\right| \le 6\ exp\left(\frac{-\lambda^2}{2||Sf||_\infty^2}\right).$$

Then applying the inequality 5.4 for this martingale, we get

$$\left|\left\{t \in (0,1]: \sup_{m\ge1}|g_m(t)| > \lambda\right\}\right| \le 6\ exp\left(\frac{-\lambda^2}{2||Sg||_\infty^2}\right).$$

But $g_m(t) = 0$ for $m \le n$. Hence

$$\left|\left\{t \in (0,1]: \sup_{m\ge n}|g_m(t)| > \lambda\right\}\right| \le 6\ exp\left(\frac{-\lambda^2}{2||Sg||_\infty^2}\right).$$

Again $S^2 g(t) = \sum_{k=0}^{\infty} d_k^2(t) = \sum_{k=0}^{\infty}[g_{k+1}(t) - g_k(t)]^2 = \sum_{k=n+1}^{\infty} d_k^2(t) = S_n'^2 f(t)$

This gives

$$\left|\left\{t \in (0,1]: \sup_{m\ge n}|g_m(t)| > \lambda\right\}\right| \le 6\ exp\left(\frac{-\lambda^2}{2||S_n' f||_\infty^2}\right).$$

i.e.

$$\left|\left\{t \in (0,1]: \sup_{m\ge n}|f_m(t) - f_n(t)| > \lambda\right\}\right|\ exp\left(\frac{-\lambda^2}{2||S_n' f||_\infty^2}\right) \tag{5.2}$$

Clearly $\{t: |f(t) - f_n(t)| > \lambda\} \subset \left\{t: \sup_{m\ge n}|f_m(t) - f_n(t)| > \lambda\right\}$.

So, we have

$$|\{t: |f(t) - f_n(t)| > \lambda\}| \le \left|\left\{t: \sup_{m\ge n}|f_m(t) - f_n(t)| > \lambda\right\}\right|$$

Consequently,

$$|\{t : |f(t) - f_n(t)| > \lambda\}| \le 6\ exp\left(\frac{-\lambda^2}{2||S_n'^2 f||_\infty^2}\right) \tag{5.3}$$

Employing the above inequality (5.3) for $n_k$, we have

$$|\{t : |f(t) - f_n(t)| > \lambda\}| \le 6\ exp\left(\frac{-\lambda^2}{2||S_n'^2 f(t)||_\infty^2}\right)$$

Note that

$$\begin{aligned} S_{nk}'^2 f(t) &= \sum_{j=n_k+1}^{\infty}[f_j(t) - f_{j+1}(t)]^2 \\ &= \sum_{j=n_k+1}^{\infty}[a_j r_j(t)]^2 = \sum_{j=n_k+1}^{\infty} a_j^2 \end{aligned}$$

This gives:

$$|\{t : |f(t) - f_n(t)| > \lambda\}| \le 6\ exp\left(\frac{-\lambda^2}{2\sum_{j=n_k+1}^{\infty} a_j^2}\right)$$

Let $\in > 0$. Then set $\lambda = \sqrt{(1+\epsilon)^2 \frac{2}{\theta^k}\log\log\theta^k}$

With this $\lambda$, the above inequality gives:

$$\left|\left\{t : |f(t) - f_{nk}(t)| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}\right\}\right|$$

$$\le 6\exp\left(\frac{-(1+\epsilon)^2\frac{2}{\theta^k}\log\log\theta^k}{2\sum_{j=n_k+1}^{\infty} a_j^2}\right)$$

Using $S_{nk}'^2 f(t) = \sum_{j=n_k+1}^{\infty} a_j^2 < \frac{1}{\theta^k}$, we get

$$\left|\left\{t : |f(t) - f_{nk}(t)| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}\right\}\right| \le 6\exp\left(\frac{-(1+\epsilon)^2\frac{2}{\theta^k}\log\log\theta^k}{\frac{1}{\theta^k}}\right)$$

$$= 6\,\frac{1}{(\log\theta)^{(1+\varepsilon)^2}}\,\frac{1}{k^{(1+\varepsilon)^2}}$$

This can be done for every $n_k$. So taking summation over all *k*, we get

$$\sum_{k=1}^{\infty}\left|\left\{t : |f(t) - f_{nk}(t)| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}\right\}\right| < \sum_{k=1}^{\infty} 6\,\frac{1}{(\log\theta)^{(1+\varepsilon)^2}}\,\frac{1}{k^{(1+\varepsilon)^2}}$$

$$= 6\,\frac{1}{(\log\theta)^{(1+\varepsilon)^2}}\sum_{k=1}^{\infty}\frac{1}{k^{(1+\varepsilon)^2}} < \infty.$$

We now invoke Borel-Cantelli Lemma. Then for a.e. $t$, there exists N large enough depending on $t$ such that for all $k \ge N$, we get,

(5.4) $$|f(t) - f_{nk}(t)| \le (1+\epsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k} = (1+\epsilon)\sqrt{\theta}\sqrt{\frac{2}{\theta^{k+1}}\log\log\theta^k}$$

Fix $t$. By definition of $n_k$, we have $S_{nk}'^2 f(t) < \frac{1}{\theta^k}$.

This gives $\theta^k < \frac{1}{S_{nk}'^2 f(t)}$.

Again, we have $S_{nk+1}'^2 f(t) < \frac{1}{\theta^{k+1}}$.

But $n_k < n_{k+1}$. Consequently, we have $S_{nk}'^2 f(t) \ge \frac{1}{\theta^{k+1}}$.

Finally, we have

(5.5) $$\frac{1}{\theta^{k+1}} \le S_{nk}'^2 f(t) < \frac{1}{\theta^k}$$

Then using (5.4) in (5.5), we have

$$|f(t) - f_{nk}(t)| \le (1+\epsilon)\sqrt{\theta}\sqrt{2.S_{nk}'^2 f(t)\ \log\log\left(\frac{1}{S_{nk}'^2 f(t)}\right)}.$$

Thus for a.e. $t$ we have

$$\frac{|f(t) - f_{nk}(t)|}{\sqrt{2.S_{nk}'^2 f(t)\ \log\log\left(\frac{1}{S_{nk}'^2 f(t)}\right)}} < (1+\varepsilon)\sqrt{\theta}.$$

This can be done for all $\varepsilon > 0$, we have

$$\frac{|f(t) - f_{nk}(t)|}{\sqrt{2.S_{nk}'^2 f(t)\ \log\log\left(\frac{1}{S_{nk}'^2 f(t)}\right)}} < \sqrt{\theta}.$$

Here as $n$ approaches to $\infty$, we have $k$ also approaches to $\infty$. If we let $\theta \searrow 1$, we have

$$\limsup_{k\to\infty}\frac{|f(t) - f_{nk}(t)|}{\sqrt{2.S_{nk}'^2 f(t)\ \log\log\left(\frac{1}{S_{nk}'^2 f(t)}\right)}} < 1$$

For almost every t. This proves our result.

## APPLICATION OF RADEMACHER FUNCTIONS

In this section, we use sequence of Rademacher functions in a law of the iterated logarithm to estimate the size of the random walks of a walker. As earlier let $\{r_k\}_{k=1}^{\infty}$ denote the sequence of Rademacher functions. Let us define:

$$f_1(t) = r_1(t)$$
$$f_2(t) = r_1(t) + r_2(t)$$
$$\vdots$$
$$f_n(t) = r_1(t) + r_2(t) + \cdots + r_n(t)$$

One can note that sum function $\{f_n(t)\}$ defines a random walk in which walker moves 1 unit to the right if $r_i(t) = 1$ and to the left if $r_i(t) = -1$. Applying a law of the iterated logarithm for this function, we have:

$$\limsup_{n\to\infty}\frac{f_n(t)}{\sqrt{2n\log\log n}} \le 1.$$

We note that for $\epsilon > 0$, this gives $|f_n(t)| \le (1+\epsilon)\sqrt{2n\log\log n}$ for sufficiently large n. But the worst bound for the function $f_n(t)$ is n i.e. $|f_n(t)| \le n$. This shows that the law of the iterated logarithm gives the sharper asymptotic estimate i.e. the estimate $|f_n(t)| \le (1+\epsilon)\sqrt{2n\log\log n}$ where for sufficiently large n, the factor $\sqrt{2n\log\log n}$ is much smaller than n. From this discussion, we can conclude that in the long run the walker will fluctuate in between $-\sqrt{2n\log\log n}$ and $\sqrt{2n\log\log n}$.

## CONCLUSIONS

We discussed the origin of the law of the iterated logarithm and the various directions where the law of the iterated logarithm has been developed. We focused on the regular and tail law of the iterated logarithm in all existing cases. As a main result, we derived a law of the iterated logarithm for the sums of Rademacher functions considering the tail sums. We expect to obtain the similar law of the iterated logarithm for tail sums in the other existing cases also which is our future direction of research.

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## AUTHOR CONTRIBUTIONS

Both the authors contributed equally in the preparation of the article.

## CONFLICT OF INTERESTS

The authors declare no conflict of interests.

## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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