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RESEARCH PAPER

A NEW VARIANT OF NEWTON'S METHOD WITH FOURTH–ORDER CONVERGENCE

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ABSTRACT

In this paper, we present new iterative method for solving nonlinear equations with fourth-order convergence. This method is free from second and higher order derivatives. We find this iterative method by using Newton's theorem for inverse function and approximating the indefinite integral in Newton's theorem by the linear combination of harmonic mean rule and Wang formula. Numerical examples show that the new method competes with Newton method, Weerakoon - Fernando method and Wang method.

Keywords: Newton method, Nonlinear equations, Iterative method, Order of convergence, Harmonic mean method

INTRODUCTION

Finding zeros of the single variable nonlinear equations efficiently is an interesting and very old problem in numerical analysis and has many applications in various branches of science and engineering. Most of the time, it is not possible to solve these equations analytically. This indicates that iterative methods are employed to get approximate solutions of nonlinear equations.

One of the most widely used iterative method is Newton method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \quad \text{.......(1)}$$

for finding numerical solution of nonlinear equation

$$f(x) = 0 \quad \text{.......(2)}$$

where $f : D \subset R \to R$ for an open interval D is a scalar function. This is an important and basic method (Bradie 2007), which converges quadratically. In the recent years, tremendous variants of this method have appeared showing one or the other advantages over this method in some sense.

$\ddot{O}$ZBAN (2004) used the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \quad \text{.......(3)}$$

and approximated the integral by harmonic mean method that is

$$\int_{x_n}^{x} f'(t)dt = (x - x_n)\frac{2f'(x)f'(x_n)}{[f'(x)+f'(x_n)]} \quad \text{.......(4)}$$

Then from (3) obtained the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)[f'(x_n)+f'(x_n^*)]}{2f'(x_n)f'(x_n^*)} \quad \text{.......(5)}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

Also Wang (2011) used the formula

$$\int_{x_n}^{x} f'(t)dt = (x - x_n)\left[(1-\beta)f'(x_n) + \beta f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)\right], \beta \neq 0 \quad \text{.......(6)}$$

and obtained the method

$$x_{n+1} = x_n - \frac{f(x_n)}{\left[(1-\beta)f'(x_n)+\beta f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)\right]} \quad \text{.......(7)}$$

THE METHOD BASED ON INVERSE FUNCTION

In this section, we use (Jain 2013), $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$, we obtain

$$g(y) = g(y_n) + \int_{y_n}^{y} g'(t)dt \quad \text{.......(8)}$$

If we approximate the indefinite integral in above equation by harmonic mean rule, we get

$$\int_{y_n}^{y} g'(t)dt = (y - y_n)\frac{2g'(y)g'(y_n)}{[g'(y)+g'(y_n)]} \quad \text{.......(9)}$$

Also from formula (6)

$$\int_{y_n}^{y} g'(t)dt = (y - y_n)\left[(1-\beta)g'(y_n) + \beta g'\left(y_n - \frac{g(y_n)}{2\beta g'(y_n)}\right)\right], \beta \neq 0 \quad \text{.......(10)}$$

If we approximate the indefinite integral of (8) by linear combination of harmonic mean rule and Wang rule, we get

$g(y) = g(y_n) + (y - y_n)\left[(1-\theta)\frac{2g'(y)g'(y_n)}{[g'(y)+g'(y_n)]} + \theta\left\{(1-\beta)g'(y_n) + \beta g'\left(y_n - \frac{g(y_n)}{2\beta g'(y_n)}\right)\right\}\right]$ ......(11)

where $y_n = f(x_n)$ and θ is any real number. Now using the fact that $g'(y) = (f^{-1})'(y) = [f'(x)]^{-1}$ and that $y = f(x) = 0$, we get the method

$$x_{n+1} = x_n - (1-\theta)\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} + \theta f(x_n)\left[\frac{(1-\beta)}{f'(x_n)} + \frac{\beta}{f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)}\right] \quad ....(12)$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

CONVERGENCE ANALYSIS

We shall prove here that the order of convergence of each method of family of the method (12) is at least three and in particular case we get fourth order method. We begin with the following.

Theorem: If a function f has sufficient number of derivative in the neighborhood ofα, which is a simple zero of f that is $f(\alpha) = 0$, $f'(\alpha) \neq 0$. Then method (12) is of order at least 3 and for unique values of $\theta = \frac{5}{3}$ and $\beta = \frac{5}{8}$, it gives a fourth order method.

Proof: Let α be a simple zero of f and e_n be the error in the nth iteration. Then, we have $x_n = \alpha + e_n$. From Taylor's series about α, we obtain

$$f(x_n) = f(\alpha + e_n)$$

$$= e_n f'(\alpha) + \frac{e_n^2}{2!}f''(\alpha) + \frac{e_n^3}{3!}f'''(\alpha) + O(e_n^4) \quad ... (13)$$

$$f'(x_n) = f'(\alpha + e_n)$$

$$= f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!}f'''(\alpha) + O(e_n^3) \quad ....(14)$$

Using $c_k = \frac{f^k(\alpha)}{k!\, f'(\alpha)}$, $k = 2, 3, 4 ...$ and from equation (13) and (14), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) \quad ..(15)$$

After some simplification, we get

$$f(x_n)[f'(x_n) + f'(x_n^*)] = 2(f'(\alpha))^2[e_n + 2c_2 e_n^2 + \left(2c_2^2 + \frac{5}{2}c_3\right)e_n^3 + O(e_n^4)] \quad ....(16)$$

$$f(x_n)f'(x_n^*) = \left(f'(\alpha)\right)^2 [1 + 2c_2 e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O(e_n^4)] \quad ... (17)$$

$$f'(x_n) + f'(x_n^*) = 2f'(\alpha)[1 + 2c_2 e_n + \left(c_2^2 + \frac{3}{2}c_3\right)e_n^2 + 2(c_2c_3 - c_2^3 + c_4)e_n^3 + O(e_n^4)] \quad ....(18)$$

$$f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right) = f'(\alpha)[1 + 2c_2(1 - \frac{1}{2\beta})e_n + \left(c_2^2\beta + 3c_3(1 - \frac{1}{2\beta})^2\right)e_n^2 + O(e_n^3)] \quad ....(19)$$

$$(1-\beta)f'(x_n) = f'(\alpha)(1-\beta)[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)] \quad ....(20)$$

$$\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} = e_n - \left(c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4) \quad ...(21)$$

and

$$f(x_n)\left[\frac{(1-\beta)}{f'(x_n)} + \frac{\beta}{f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)}\right] = e_n + \left[\frac{1}{\beta}c_2^2 - (2c_2^2 - c_3) - \frac{3}{4\beta}c_3\right]e_n^3 + O(e_n^4) \quad ....(22)$$

Thus, from (12)

$$\begin{aligned} e_{n+1} &= e_n - (1-\theta)\left[e_n - \left(c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4)\right] \\ &\quad -\theta\left[e_n + \left\{\frac{1}{\beta}c_2^2 - (2c_2^2 - c_3) - \frac{3}{4\beta}c_3\right\}e_n^3 + O(e_n^4)\right] \\ &= (1-\theta)\left(c_2^2 + \frac{1}{2}c_3\right)e_n^3 + \theta\left[-\frac{1}{\beta}c_2^2 + (2c_2^2 - c_3) + \frac{3}{4\beta}c_3\right]e_n^3 + O(e_n^4). \quad .... (23) \end{aligned}$$

For unique values of $\theta = \frac{5}{3}$ and $\beta = \frac{5}{8}$, from equation (23), we get $e_{n+1} = O(e_n^4)$. Thus order of convergence of family of method (23) is at least three and it gives the fourth order method for unique value of $\theta = \frac{5}{3}$ and $\beta = \frac{5}{8}$. The new forth order method is

$$x_{n+1} = x_n + \frac{4}{3}\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} + \frac{5}{24}f(x_n)\left[\frac{3}{f'(x_n)} + \frac{5}{f'\left(x_n - \frac{4f(x_n)}{5f'(x_n)}\right)}\right] \quad ....(24)$$

Where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

NUMERICAL EXPERMENTS

In order to check the performance of the introduced fourth-order method, we give the numerical results on some test functions. We also compare the results of this method with Newton method (NM), Weerakoon and Fernando (W-F) method and Wang Method. Numerical computations have been performed using the Matlab software rounding to 16 significant decimal digits. We use the stopping criteria $|x_{n+1} - x_n| < \varepsilon$ where $\varepsilon = (10)^{-8}$ for the iterative process of our results.

The test functions and their roots α which are used as numerical examples are given below:

(i) $f_1 = cosx - xe^x + x^2$, $\alpha = 0.639154069332008$

(ii) $f_2 = (x-1)^8 - 1$, $\alpha = 2$

(iii) $f_3 = x^3 + 4x^2 - 10$, $\alpha = 1.365230013414007$

Table 1: $f_1 = cosx - xe^x + x^2$ and initial guess $x = 1$

n	Newton method	W-F method	Wang method	Present method(24)
1	0.724644697567095	0.665881945014898	0.662938687425796	0.645000328957045
2	0.644658904870270	0.639169572742496	0.639163361765903	0.639154096912870
3	0.639177807467281	0.639154096332011	0.639154096332008	
4	0.639154096773051			

Table 2: $f_2 = (x-1)^8 - 1$, and initial guess $x = 2.5$

n	Newton method	W-F method	Wang method	Present method(24)
1	2.319815957933242	2.244107083493465	2.236236164659438	2.177268585515064
2	2.172758071196855	2.068965943835277	2.060772262760990	2.014981732655870
3	2.067131900591505	2.003572340012848	2.002277983579724	2.000002165842810
4	2.013060520391305	2.000000704582805	2.000000159604880	2.000000000000000
5	2.000574370391506	2.000000000000000	2.000000000000000	
6	2.000001152667968			
7	2.000000000004650			

Table 3: $f_3 = x^3 + 4x^2 - 10$ and initial guess $x = 1$

n	Newton method	W-F method	Wang method(2011)	Present method(24)
1	1.454545454545455	1.345024237239806	1.346506300114548	1.369968652351256
2	1.368900401069519	1.365227728691384	1.365228321128059	1.365230013487738
3	1.365236600202116	1.365230013414097	1.365230013414097	
4	1.365230013435367			

Table 4: Comparison

Function	Newton method		W-F method		Wang method(2011)		Present method (24)	
	TNFE	TNI	TNFE	TNI	TNFE	TNI	TNFE	TNI
f_1	8	4	9	3	9	3	8	2
f_2	14	7	15	5	15	5	16	2
f_3	8	4	9	3	9	3	8	2

TNFE = Total number of function evaluation, TNI = Total number of iteration.

CONCLUSION

Method (12) is family of at least third order methods which contains Weerakoon and Fernando method, inverse Wang method as well as many other third order methods. For the unique value of θ and β it gives the fourth order method which is actually linear combination of two third order methods, inverse harmonic mean and inverse

Wang. The numerical experiment results show that new introduced fourth-order method can easily compete with classical Newton method, Wang method as well as Weerakoon and Fernando method. Also this method does not require the computation of second or higher order derivatives.

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