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Bayesian Estimation Using Progressively Censored Masked Data Under Asymmetric Loss Function

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ABSTRACT

Competing risk modeling is very useful for the assessment of component characteristics in reliability studies. In this paper, we consider the competing risk modeling of progressively censored data when units under lifetest are series system of two components. Assuming the lifetime distributions of components to be exponentially distributed, we obtain Bayes estimate of parameters and components relative risks under asymmetric loss functions. Bayesian computation is done using Lindley's approximation. A simulation study is presented for numerical illustrations.

Keywords: Asymmetric Loss function, Bayes estimate, Competing risk, Masked data, Relative risk.

INTRODUCTION

Lifetime data are often analyzed under competing risk model when the units under test are exposed to more than one mutually exclusive causes of failure and failure due to one cause excludes the chances of failure due to other causes (Crowder 2001). For example, computer system may fail due to any of its components like CPU, Monitor or Power supply, which are connected in series. If we want to assess the lifetime characteristics of any particular components, the theory of competing risk may be useful. In the life testing experiments conducted with series systems, one can obtain the data regarding the lifetimes of the systems along with their respective causes of failure. It may also happen sometimes that we may not be able to diagnose the causes of failures for some of the failed units. Such data are termed as masked data, that is, the data where the cause of failure of some of the failed systems are missing (Flehtnger *et al.* 1998).

Many authors have considered the analysis of competing risk data with missing cause of failure. Miyakawa (1984) considered the analysis of masked data and obtained the maximum likelihood estimators (MLE's) of two component series systems of exponential components. Usher and Hodgson (1988) discuss the exact and partial masking problem for three component of exponential distribution. Mukhopadhayay and Basu (1993) considered the analysis of masked series system life time data from exponential distribution. Mukhopadhyay and Basu (1997) presented the analysis of k-independent Weibull components with equal shape parameters. Xu and Tang (2009) considered the component reliability analysis with Pareto reliability models. Singh and Tomer (2011) obtained the MLEs of component reliabilities using incomplete competing risk data. Cramer and Schmiedt (2011) discussed the analysis of progressively type-II censored competing risk data of Lomax distribution. Tomer *et al.* (2013) presented the analysis of masked series system life time data from a generalized lifetime distribution.

Censoring is very important feature of life testing experiments. Plenty of censoring schemes have been developed for lifetime experiment in past decade (Lawless 2003). Here we considered progressive type-II censoring scheme which is very popular among the academician working in the field of reliability and lifetimes studies (Balakrishnan & Aggarwala 2000). The progressive type-II censoring is described as follows.

Let the random variable X denotes the lifetime of a unit. Suppose that n identical units are put to test and non-negative integers $R_1, R_2, ..., R_m$ are fixed in advance satisfying $R_1 + R_2 + ... + R_m = n - m$. At the time of first failure, R_1 of the remaining $n-1$ units are randomly removed. At the time of second failure, R_2 units out of the remaining $n - 2 - R_1$ units are randomly removed and so on. Finally, at the time of m^{th} failure the experiment is terminated by removing all remaining $R_m = n - R_1 + R_2 + ... + R_{m-1} - m$ units.

In this paper, we discuss the analysis of progressively type-II censored data under competing risk model. We consider the situation when cause of failure for some of the systems is missing and provide ML and Bayes estimates of reliability and relative risk rate under asymmetric loss function. We derive the likelihood function and obtain MLEs of parameters, reliability functions and relative risk rates. We also obtain Bayes estimates of these parametric functions under squared error as well as LINEX loss functions. Finally, we carry out simulation study and conclude the findings.

MAXIMUM LIKELIHOOD ESTIMATION

Let X_1 and X_2 respectively denotes the lifetimes of component 1 and component 2 of a two-component series system. The probability density function (*pdf*) and the reliability function, at a mission time t_o, of $X_j, j=1,2$ are given respectively, by

$$f_j(x)=\frac{1}{\lambda_j}\exp\left(-\frac{x}{\lambda_j}\right) \tag{1}$$

and $$\overline{F}(x;\lambda_j)=\exp\left(-\frac{x}{\lambda_j}\right). \tag{2}$$

Suppose, in an experiment *n* identical systems are put to test and the progressive type-II censoring scheme is followed. After the termination of test, lifetimes of *m* systems $X_{1:m}, X_{2:m},.....,X_{m:m}$, along with the cause of failure of all the failed systems, are observed. The observed data set is denoted by $\underline{X}=\{X_{i:m},\delta_i\}, i=1,2,...,m,$ where $X_{i:m}=\min\{X_{1i},X_{2i}\}$ and δ_i is cause of failure of i^{th} system. Let out of m failures, m_1 and m_2, respectively denote the number of failure occur due to the failure of component 1 and component 2. Further, m_{12} is the number of systems for which the cause of failure is missing. It is evident that $m_1+m_2+m_{12}=m$. Throughout the rest part of the paper, we use notation X_i instead $X_{i:m}$. With these notations, we write the likelihood function of the observed data as follows.

$$L(\lambda_1,\lambda_2\,|\,\underline{x})\propto\prod_{i=1}^{m_1} f(x_i;\lambda_1)\overline{F}(x_i;\lambda_2)\prod_{i=1}^{m_2} f(x_i;\lambda_2)\overline{F}(x_i;\lambda_1) \tag{3}$$
$$\prod_{i=1}^{m}\left\{\overline{F}(x_i;\lambda_1)\overline{F}(x_i;\lambda_2)\right\}^{R_i}.$$

Using (1) and (2), we get from (3) that

$$L(\lambda_1,\lambda_2\,|\,\underline{x})=\prod_{i=1}^{m_1}\left[\frac{1}{\lambda_1}\exp\left\{-x_i\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right\}\right]\prod_{i=1}^{m_2}\left[\frac{1}{\lambda_2}\exp\left\{-x_i\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right\}\right]$$

$$\prod_{i=1}^{m_{12}}\left[\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\exp\left\{-x_i\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right\}\right]\prod_{i=1}^{m}\left[\exp\left\{-x_iR_i\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right\}\right]$$

$$=\frac{1}{\lambda_1^{m_1}\lambda_2^{m_2}}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^{m_{12}}\exp\left\{-\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\sum_{i=1}^{m}(R_i+1)x_i\right\} \tag{4}$$

Taking the logarithm of (4) and differentiating it with respect to λ_1 and λ_2, respectively, we get the following likelihood equations.

$$\lambda_1\lambda_2 m_{12}+(\lambda_1+\lambda_2)\left(m_1\lambda_1-\sum_{i=1}^{m}(R_i+1)x_i\right)=0$$

and $$\lambda_1\lambda_2 m_{12}+(\lambda_1+\lambda_2)\left(m_2\lambda_2-\sum_{i=1}^{m}(R_i+1)x_i\right)=0.$$

Solving the above likelihood equations simultaneously, we obtain the MLEs of λ_1 and λ_2 as follows.

$$\hat{\lambda}_1=\frac{(m_1+m_2)}{m_1 m}\sum_{i=1}^{m}(R_i+1)x_i \tag{5}$$

and $$\hat{\lambda}_2=\frac{(m_1+m_2)}{m_2 m}\sum_{i=1}^{m}(R_i+1)x_i. \tag{6}$$

Using the invariance property of MLE, the ML estimate of reliability function of j^{th} component, at mission time t_o, can be obtained from (2) on substituting the value of MLE of $\lambda_j;\ j=1,2,$ are as follows $\hat{R}_j(t)=\exp\left(-\frac{t_0}{\hat{\lambda}_j}\right).$

Component relative risk rate

The relative risk rate π_1^* of the component 1 is given by

$$\pi_1^*=P(X_1<X_2)=\int_0^\infty\frac{1}{\lambda_1}\exp\left(-\frac{x}{\lambda_1}\right)\exp\left(-\frac{x}{\lambda_2}\right)dx$$
$$=\frac{\lambda_2}{\lambda_1+\lambda_2}. \tag{7}$$

The relative risk rate π_2^* of component 2 is

$$\pi_2^*=P(X_2<X_1)=\int_0^\infty\frac{1}{\lambda_2}\exp\left(-\frac{x}{\lambda_2}\right)\exp\left(-\frac{x}{\lambda_1}\right)dx$$
$$=\frac{\lambda_1}{\lambda_1+\lambda_2}=1-\pi_1^*. \tag{8}$$

Using the invariance property of MLE, the MLEs of relative risk rates of component 1 and component 2 are given, respectively, by

$$\hat{\pi}_1^*=\frac{\hat{\lambda}_2}{\hat{\lambda}_1+\hat{\lambda}_2}\ \textit{and}\ \hat{\pi}_2^*=\frac{\hat{\lambda}_1}{\hat{\lambda}_1+\hat{\lambda}_2}$$

Bayesian Estimation

In Bayesian paradigm, we consider λ_1 and λ_2 to be random variables. Let us assume that $\lambda_j, j=1,2.$ follow inverted gamma density with parameter μ_j and ν_j given by,

$$p(\lambda_j)=\frac{\mu_j^{\nu_j}}{\Gamma\nu_j}\frac{1}{\lambda_j^{\nu_j+1}}\exp\left(-\frac{\mu_j}{\lambda_j^2}\right);\ j=1,2. \tag{9}$$

Considering λ_1 and λ_2 independent and merging the likelihood function (4) with joint prior distribution of λ_1 and λ_2, we obtain the joint posterior density of λ_1 and λ_2 as follows.

$$\pi(\lambda_1,\lambda_2 \mid \underline{x}) = K\frac{1}{\lambda_1^{m_1+\nu_1+1}\lambda_2^{m_2+\nu_2+1}}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^{m_{12}} \tag{10}$$

$$xp\left\{-\frac{1}{\lambda_1}\left(\mu_1+\sum_{i=1}^{m}(R_i+1)x_i\right)\right\}\exp\left\{-\frac{1}{\lambda_2}\left(\mu_2+\sum_{i=1}^{m}(R_i+1)x_i\right)\right\}$$

Where, $K^{-1}=\int_0^\infty\int_0^\infty L(\lambda_1,\lambda_2 \mid \underline{x})p(\lambda_1,\lambda_2)d\lambda_1\, d\lambda_2.$

From (10), we observe that the marginal distribution of λ_1 and λ_2 cannot be obtained in closed form, which is essential in order to obtain Bayes estimates of individual parametric functions. Therefore in order to get Bayes estimates of parameters, we proceed with Lindley approximation.

According to Lindley's (1980) approximation, the posterior expectation of any parametric function $\omega(\lambda)=\omega(\lambda_1,\lambda_2)$ which is a ratio of two integrals, given by

$$E(\omega(\lambda)\mid \underline{x})=\frac{\int \omega(\lambda)\, L(\lambda_1,\lambda_2 \mid \underline{x})p(\lambda_1,\lambda_2)d\lambda_1\, d\lambda_2}{\int L(\lambda_1,\lambda_2 \mid \underline{x})p(\lambda_1,\lambda_2)d\lambda_1\, d\lambda_2},$$

can be obtained in the form of the following expression.

$$\tilde{\omega}=\hat{\omega}(\lambda)+\frac{1}{2}\left[A+l_{30}B_{12}+l_{03}B_{21}+l_{21}C_{12}+l_{12}C_{21}\right]+\rho_1 A_{12}+\rho_2 A_{21} \tag{11}$$

where, $A=\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\sigma_{ij}$; $l_{\eta\xi}=\frac{\partial^{\eta+\xi}l}{\partial\lambda_1^{\eta}\partial\lambda_2^{\xi}}$, $l=\log(L)$,

$\eta\ \&\ \xi=0,1,2,3;\quad \eta+\xi=3 \quad for\ i,j=1,2.$ $\rho_i=\frac{\partial\rho}{\partial\lambda_i}$,

$\omega_i=\frac{\partial\omega}{\partial\lambda_i}$, $\omega_{ij}=\frac{\partial^2\omega}{\partial\lambda_i\partial\lambda_j}$, where $\rho=\log\pi(\lambda_1,\lambda_2)$ and

$for\ i\neq j$, $A_{ij}=\omega_i\sigma_{ii}+\omega_j\sigma_{ji}$, $B_{ij}=(\omega_i\sigma_{ii}+\omega_j\sigma_{ij})\sigma_{ii}$

and $C_{ij}=3\omega_i\sigma_{ii}\sigma_{ij}+\omega_j(\sigma_{ii}\sigma_{jj}+2\sigma_{ij}^2)$. Here σ_{ij} is the $(i,j)^{th}$ element in the inverse of the matrix $\{-l_{ij}\}, i,j=1,2,$ such that $l_{ij}=\frac{\partial^2 l}{\partial\lambda_i\partial\lambda_j}$.

Let $\sigma_{11}=\frac{H}{N},\quad \sigma_{22}=\frac{G}{N},\quad \sigma_{12}=\sigma_{21}=-\frac{I}{N}$,

where $N=GH-I^2$ with these notations, using the expression given by Nassar and Eissa (2004), (11) can be written as follows,

$$E(\omega(\lambda)\mid d)=\hat{\omega}(\lambda)+\omega_1\psi_1+\omega_2\psi_2+\phi, \tag{12}$$

Where $\psi_1=\frac{1}{N}(H\rho_1-I\rho_2)+\frac{1}{2N^2}[H^2l_{30}-IGl_{03}+(GH+2I^2)l_{12}-3IHl_{21}]$,

$\psi_2=\frac{1}{N}(G\rho_2-I\rho_1)+\frac{1}{2N^2}[G^2l_{03}-IHl_{30}+(GH+2I^2)l_{21}-3IGl_{12}]$

and $\phi=\frac{1}{2N}[H\omega_{11}-I(\omega_{12}+\omega_{21})+G\omega_{22}]$.

Using the notation $\chi=\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)$, and $T=\sum_{i=1}^{m}(R_i+1)x_i$

We derive the expressions for our estimation problem.

$$G=\frac{m_{12}}{\lambda_1^4\chi^2}(2\lambda_1\chi-1)+\frac{m_1}{\lambda_1^2}-\frac{2T}{\lambda_1^3},$$

$$H=\frac{m_{12}}{\lambda_2^4\chi^2}(2\lambda_2\chi-1)+\frac{m_2}{\lambda_2^2}-\frac{2T}{\lambda_2^3},\ I=\frac{-m_{12}}{\lambda_1^2\lambda_2^2\chi^2},$$

$$l_{12}=l_{21}=\frac{2m_{12}}{\lambda_1^2\lambda_2^4\chi^3}(\lambda_2\chi-1),$$

$$l_{30}=\frac{2m_{12}}{\lambda_1^6\chi^3}(-3\lambda_1^2\chi^2+3\lambda_1\chi-1)-\frac{2m_1}{\lambda_1^3}+\frac{6T}{\lambda_1^4}\ \text{and}$$

$$l_{03}=\frac{2m_{12}}{\lambda_2^6\chi^3}(-3\lambda_2^2\chi^2+3\lambda_2\chi-1)-\frac{2m_2}{\lambda_2^3}+\frac{6T}{\lambda_2^4}.$$

Bayes estimates Under SELF

Consider Square Error Loss Function (SELF) given by $L(\theta,\hat{\theta})=(\hat{\theta}-\theta)^2$, where $\hat{\theta}$ is the estimator of unknown parameter θ. Bayes estimator of any parameter is its posterior mean, we obtain the Bayes estimates of λ_1,λ_2, component reliabilities and relative risk rates by using (12) as follows.

(i) If $\omega=\lambda_1$, we have $\omega_1=1,\ \omega_2=0$ and $\phi=0$. Substituting these values in (12), we get the following Bayes estimator of λ_1

$$\tilde{\lambda}_{1S}=\hat{\lambda}_1+\psi_1. \tag{13}$$

(ii) If $\omega=\lambda_2$, we have $\omega_2=1,\ \omega_1=0$ and $\phi=0$. Substituting these values in (12), the Bayes estimator of λ_2 comes out to be

$$\tilde{\lambda}_{2S}=\hat{\lambda}_2+\psi_2. \tag{14}$$

(iii) If $\omega=R_1(t)=\exp(-t/\lambda_1)$, we have

$$\tilde{R}_{1S}(t)=\hat{R}_1(t)+\omega_1\psi_1+\phi_{R1}, \tag{15}$$

where $\omega_1=t/\lambda_1^2\exp(-t/\lambda_1)$ and $\phi_{R_1}=Ht(t-2\lambda_1)\exp(-t/\lambda_1)/(2N\lambda_1^4)$.

(iv) If $\omega=R_2(t)=\exp(-t/\lambda_2)$, we have

$$\tilde{R}_{2S}(t)=\hat{R}_2(t)+\omega_2\psi_2+\phi_{R2}, \tag{16}$$

where $\omega_2 = t/\lambda_2^2 \exp(-t/\lambda_2)$ and

$\phi_{R_2} = Gt(t-2\lambda_2)\exp(-t/\lambda_2)/(2N\lambda_2^4)$.

(v) To get the Bayes estimate of relative risk rate of component 1, we have $\omega = \lambda_2/(\lambda_1+\lambda_2)$ and get the following expression

$$\tilde{\pi}_{1S}^* = \hat{\pi}_1^* + \omega_1\psi_1 + \omega_2\psi_2 + \phi_{RR_1}, \quad (17)$$

where, $\phi_{RR_1} = \frac{1}{2N}(H\omega_{11} - I(\omega_{21}+\omega_{12}) + G^*\omega_{22})$,

$\omega_1 = -\lambda_2/(\lambda_1+\lambda_2)^2$, $\omega_{11} = 2\lambda_2/(\lambda_1+\lambda_2)^3$, $\omega_2 = \lambda_1/(\lambda_1+\lambda_2)^2$, $\omega_{22} = -2\lambda_1/(\lambda_1+\lambda_2)^3$ and $\omega_{12} = (\lambda_2-\lambda_1)/(\lambda_1+\lambda_2)^3$.

(vi) Similarly, the Bayes estimate of relative risk rate of component 2, we have $\omega = \lambda_1/(\lambda_1+\lambda_2)$ and obtain the expression

$$\tilde{\pi}_{2S}^* = \hat{\pi}_2^* + \omega_1\psi_1 + \omega_2\psi_2 + \phi_{RR_2}, \quad (18)$$

where, $\phi_{RR_{21}} = \frac{1}{2N}(H\omega_{11} - I(\omega_{21}+\omega_{12}) + G^*\omega_{22})$,

$\omega_2 = -\lambda_1/(\lambda_1+\lambda_2)^2$, $\omega_{22} = 2\lambda_1/(\lambda_1+\lambda_2)^3$, $\omega_1 = \lambda_2/(\lambda_1+\lambda_2)^2$, $\omega_{11} = -2\lambda_2/(\lambda_1+\lambda_2)^3$ and $\omega_{12} = (\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)^3$.

Bayes estimates Under LINEX loss function

Varian (1975) introduced the convex loss function known as is LINEX (Linear-exponential) loss function, defined as follows $L(\hat{\theta},\theta) = be^{c\Delta} - c\Delta - b;\ c \neq 0\ , b > 0$

The Bayes Estimate of any parametric function θ under LINEX loss function (Zellner 1986) comes out to be

$$\tilde{\theta}_L = -\frac{1}{c}[E_\theta(\exp(-c\theta))], \quad (19)$$

Provided $E_\lambda(\exp(-c\theta))$ exist and finite.

To obtain $\tilde{\theta}_L$, the Bayes estimate of θ under LINEX loss function, we first find the posterior expectation $E_\theta(\exp(-c\theta))$ for given c, by using (12). Bayes estimates of λ_1, λ_2, component reliabilities and relative risk rate can be obtained as follows.

(i) When $\omega = \exp(-c\lambda_1)$, using (12) and (19), we have the Bayes estimate of λ_1

$$\tilde{\lambda}_{1L} = -\frac{1}{c}\log E_\alpha[(\exp(-c\hat{\lambda}_1))]$$

$$= -\frac{1}{c}\log[\exp(-c\lambda_1) + \omega_1\psi_1 + \phi_{\hat{\lambda}_{1L}}] \quad (20)$$

where $\omega_1 = -c\exp(-c\lambda_1)$ and $\phi_{\hat{\lambda}_{L1}} = \frac{c^2 H\exp(-c\lambda_1)}{2N}$.

(ii) Similarly when $\omega = \exp(-c\lambda_2)$, the Bayes estimate of λ_2 is given by

$$\tilde{\lambda}_{2L} = -\frac{1}{c}\log[\exp(-c\lambda_2) + \omega_2\psi_2 + \phi_{\hat{\lambda}_{2L}}] \quad (21)$$

where $\omega_2 = -c\exp(-c\lambda_2)$ an $\phi_{\lambda_{2L}} = \frac{c^2 G\exp(-c\lambda_2)}{2N}$.

(iii) When $\omega = \exp\{-c\exp(-t/\lambda_1)\}$, the Bayes estimate of the reliability function of component-1 comes out to be

$$\tilde{R}_{1L}(t) = -\frac{1}{c}\log E_R[\exp(-c\exp(-t/\lambda_1))]$$

$$= -\frac{1}{c}\log[\exp\{-c\exp(-t/\lambda_1)\} + \omega_1\psi_1 + \phi_{1L}], \quad (22)$$

where, $\omega_1 = -\frac{ct}{\lambda_1^2}\exp(-t/\lambda_1)\exp\{-c\exp(-t/\lambda_1)\}$

$\phi_{1L} = \frac{Hct}{2N\hat{\lambda}_1^4}\exp(-t/\lambda_1)\exp\{-c\exp(-t/\lambda_1)\}\{2\lambda_1 - t + ct\exp(-t/\lambda_1)\}$

(iv) When $\omega = \exp\{-c\exp(-t/\lambda_2)\}$, the Bayes estimate of reliability function component-2 is

$$\tilde{R}_{2L}(t) = -\frac{1}{c}\log E_R[\exp(-c\exp(-t/\lambda_2))]$$

$$= -\frac{1}{c}\log[\exp\{-c\exp(-t/\lambda_2)\} + \omega_2\psi_2 + \phi_{2L}], \quad (23)$$

where, $\omega_2 = -\frac{ct}{\lambda_2^2}\exp(-t/\lambda_2)\exp\{-c\exp(-t/\lambda_2)\}$,

$\phi_{2L} = \frac{Gct}{2N\lambda_2^4}\exp(-t/\lambda_2)\exp\{-c\exp(-t/\lambda_2)\}\{2\lambda_2 - t + ct\exp(-t/\lambda_2)\}$

(v) when $\omega = \exp\{-c\lambda_2/(\lambda_1+\lambda_2)\}$, the Bayes estimate of relative risk rate of component 1 is given by

$$\tilde{\pi}_{1L}^*(t) = -\frac{1}{c}\log E_R[\exp\{-c\lambda_2/(\lambda_1+\lambda_2)\}]$$

$$= -\frac{1}{c}\log[\exp\{-c\lambda_2/(\lambda_1+\lambda_2)\} + \omega_1\psi_1 + \omega_2\psi_2 + \phi_{3L}], \quad (24)$$

Where $\phi_{3L} = \frac{1}{2N}[H\omega_{11} - I(\omega_{12}+\omega_{21}) + G\omega_{22}]$. and let $\tau_1 = \exp\{-c\lambda_2/(\lambda_1+\lambda_2)\}$ so that

$\omega_1 = c\tau_1\lambda_2/(\lambda_1+\lambda_2)^2$,

$\omega_{11} = c\tau_1\lambda_2\{\lambda_2(c-2) - 2\lambda_1\}/(\lambda_1+\lambda_2)^4$,

$\omega_2 = -c\tau_1\lambda_1/(\lambda_1+\lambda_2)^2$

$\omega_{22} = c\tau_1\lambda_1\{\lambda_1(c-2) + 2\lambda_2\}/(\lambda_1+\lambda_2)^4$ and

$\omega_{12} = c\tau_1\{\lambda_1^2 - \lambda_2^2 - c\lambda_1\lambda_2\}/(\lambda_1+\lambda_2)^4$.

(vi) when $\omega = \exp\{-c\lambda_1/(\lambda_1+\lambda_2)\}$, the Bayes estimate of relative risk rate of component 1 is given by

$$\tilde{\pi}^*_{2L}(t) = -\frac{1}{c}\log E_R[\exp\{-c\lambda_1/(\lambda_1+\lambda_2)\}],$$
$$= -\frac{1}{c}\log[\exp\{-c\lambda_1/(\lambda_1+\lambda_2)\} + \omega_1\psi_1 + \omega_2\psi_2 + \phi_{4L}], \quad (25)$$

where, $\phi_{4L} = \frac{1}{2N}[H\omega_{11} - I(\omega_{12}+\omega_{21}) + G\omega_{22}]$.

Further ψ_1 and ψ_2 can be evaluated using (12) with $\omega_2 = c\tau_2\lambda_1/(\lambda_1+\lambda_2)$,

$\omega_{22} = c\tau_2\lambda_1\{\lambda_1(c-2) - 2\lambda_2\}/(\lambda_1+\lambda_2)^4$,

$\omega_1 = c\tau_2\lambda_2/(\lambda_1+\lambda_2)^2$,

$\omega_{11} = c\tau_2\lambda_2\{\lambda_1(c+2) + 2\lambda_1\}/(\lambda_1+\lambda_2)^4$,

$\omega_{12} = c\tau_2\{\lambda_2^2 - \lambda_1^2 - c\lambda_1\lambda_2\}/(\lambda_1+\lambda_2)^4$ and

$\tau_2 = \exp\{-c\lambda_1/(\lambda_1+\lambda_2)\}$.

Simulation Study

Here we give some illustrations based on simulation study. For the values of $\lambda_1 = 1.8$ and $\lambda_2 = 2.0$ we generate exponential lifetimes X_1 and X_2, which are considered to be the lifetimes of component 1 and 2. We are considering series system the lifetime of systems become $X = \min(X_1, X_2)$. We generate *n* observations in this way and also note down their cause of failures. From *n* observations, we draw a sample of *m* observations following progressive type-II censoring scheme. In this sample we randomly mask the cause of failure of 30% observations and get the final form of the competing risk data with missing cause of failure.

For the generated progressively censored systems lifetime data with missing cause of failure, we obtain MLEs of component parameters, reliabilities and relative risk rates by using the derived expressions for the same. We provide these estimates under varieties of schemes by choosing several patterns of removals of observations under progressive censoring scheme.

For Bayesian estimation, we choose values of hyper parameters to be $\mu_1 = \mu_2 = 2$ *and* $\nu_1 = \nu_2 = 2.5$ and obtain Bayes estimate of distribution parameters λ_1 and λ_2, component reliabilities and relative risk rates using Lindley's approximation. We run this process for 2000 times and present the averages estimates of considered Parametric function along with their mean square errors (MSE). The average Bayes estimate and their posterior risk under different loss functions are given in Tables 1 and 2. The estimates of relative risk rates are given Tables 3 and 4 and values of estimates of components reliabilities are given in Tables 5 and 6. Finally we observe that MSEs of component reliability decreases when *n* increases and for fixed *n* the MSEs of all the estimates decreases when *m* increases.

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Table 1. Average Bayes estimates of parameters and corresponding MSEs (in Bracket) under different loss functions for n=30 and C_1=-2.5 and C_2= 3.

m	Scheme	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\tilde{\lambda}_{1S}$	$\tilde{\lambda}_{2S}$	$\tilde{\lambda}_{1LC_1}$	$\tilde{\lambda}_{2LC_1}$	$\tilde{\lambda}_{1LC_2}$	$\tilde{\lambda}_{2LC_2}$
10	$0^{*9}, 20$	1.707	1.916	1.511	1.562	1.236	1.353	2.147	2.450
		(0.354)	(0.504)	(0.168)	(0.269)	(0.422)	(0.587)	(0.572)	(0.834)
	$0^{*3}, 4^{*5}, 0^{*2}$	1.704	1.925	1.510	1.561	1.235	1.349	2.145	2.439
		(0.356)	(0.505)	(0.167)	(0.270)	(0.425)	(0.591)	(0.575)	(0.853)
	$0, (0,\ 5)^{*4}, 0$	1.485	1.650	1.391	1.438	1.110	1.192	1.899	2.130
		(0.450)	(0.613)	(0.259)	(0.406)	(0.564)	(0.792)	(0.593)	(0.853)
	2^{*10}	1.457	1.649	1.377	1.435	1.095	1.188	1.863	2.132
		(0.460)	(0.610)	(0.270)	(0.408)	(0.581)	(0.796)	(0.688)	(1.029)
	$20, 0^{*9}$	1.406	1.583	1.350	1.406	1.072	1.156	1.807	2.055
		(0.503)	(0.662)	(0.298)	(0.446)	(0.610)	(0.841)	(0.689)	(1.046)
15	$0^{*14}, 15$	1.763	1.984	1.645	1.772	1.366	1.496	2.109	2.406
		(0.293)	(0.420)	(0.163)	(0.210)	(0.301)	(0.424)	(0.434)	(0.652)
	$0^{*5}, 3^{*5}, 0^{*5}$	1.760	1.984	1.643	1.772	1.365	1.497	2.106	2.407
		(0.294)	(0.429)	(0.164)	(0.215)	(0.302)	(0.428)	(0.553)	(0.863)
	$0^{*3}, (0,\ 3)^{*5}, 0^{*2}$	1.519	1.695	1.468	1.576	1.204	1.300	1.833	2.071
		(0.365)	(0.483)	(0.251)	(0.333)	(0.455)	(0.630)	(0.452)	(0.645)
	1^{*15}	1.517	1.718	1.466	1.588	1.203	1.314	1.829	2.101
		(0.365)	(0.492)	(0.251)	(0.330)	(0.455)	(0.619)	(0.451)	(0.688)
	$15, 0^{*14}$	1.404	1.586	1.386	1.498	1.132	1.230	1.702	1.948
		(0.428)	(0.567)	(0.308)	(0.407)	(0.535)	(0.727)	(0.554)	(0.850)
20	$0^{*19}, 10$	1.806	2.012	1.720	1.864	1.453	1.580	2.107	2.379
		(0.252)	(0.360)	(0.157)	(0.200)	(0.232)	(0.334)	(0.370)	(0.551)
	$0^{*5}, 1^{*10}, 0^{*5}$	1.795	2.007	1.712	1.859	1.445	1.575	2.094	2.373
		(0.259)	(0.364)	(0.159)	(0.203)	(0.234)	(0.339)	(0.473)	(0.715)
	$0^{*5}, (0,\ 2)^{*5}, 0^{*5}$	1.559	1.745	1.526	1.659	1.279	1.389	1.828	2.074
		(0.305)	(0.421)	(0.224)	(0.299)	(0.369)	(0.516)	(0.378)	(0.572)
	$(0, 1)^{*10}$	1.636	1.826	1.586	1.721	1.332	1.446	1.915	2.166
		(0.279)	(0.388)	(0.197)	(0.261)	(0.321)	(0.455)	(0.399)	(0.598)
	$10, 0^{*19}$	1.544	1.715	1.514	1.637	1.268	1.369	1.812	2.037
		(0.308)	(0.427)	(0.228)	(0.311)	(0.378)	0.536)	(0.491)	(0.719)
25	$0^{*24}, 5$	1.819	2.032	1.754	1.919	1.510	1.647	2.073	2.348
		(0.220)	(0.320)	(0.152)	(0.201)	(0.190)	(0.276)	(0.363)	(0.520)
	$0^{*10}, 1^{*5}, 0^{*10}$	1.810	2.031	1.747	1.918	1.504	1.646	2.063	2.348
		(0.225)	(0.324)	(0.154)	(0.203)	(0.191)	(0.278)	(0.368)	(0.612)
	$0^{*5}, (0, 0, 1)^{*5}, 0^{*5}$	1.711	1.904	1.663	1.813	1.430	1.552	1.953	2.201
		(0.234)	(0.328)	(0.173)	(0.230)	(0.242)	(0.348)	(0.357)	(0.530)
	$(0, 0, 0, 0, 1)^{*5}$	1.757	1.967	1.702	1.865	1.464	1.598	2.004	2.274
		(0.223)	(0.325)	(0.160)	(0.218)	(0.216)	(0.314)	(0.353)	(0.578)
	$5, 0^{*24}$	1.699	1.898	1.653	1.810	1.421	1.547	1.938	2.195
		(0.238)	(0.328)	(0.177)	(0.230)	(0.249)	(0.350)	(0.406)	(0.617)
30	0^{*30}	1.820	2.053	1.771	1.961	1.544	1.699	2.046	2.342
		(0.203)	(0.295)	(0.149)	(0.196)	(0.166)	(0.235)	(0.353)	(0.519)

(NOTE: Here a^{*b} means ($a, a, a, \ldots, b$ times)

Table 2. Average Bayes estimates of parameters and corresponding MSEs (in Bracket) under different loss functions for n=50 and C_1=-2.5 and C_2= 3.

m	Scheme	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\tilde{\lambda}_{1S}$	$\tilde{\lambda}_{2S}$	$\tilde{\lambda}_{1LC_1}$	$\tilde{\lambda}_{2LC_1}$	$\tilde{\lambda}_{1LC_2}$	$\tilde{\lambda}_{2LC_2}$
10	$0^{*9}, 40$	1.709	1.900	1.509	1.557	1.237	1.344	2.151	2.430
		(0.360)	(0.501)	(0.168)	(0.272)	(0.423)	(0.598)	(0.568)	(0.811)
	$0, 5^{*8}, 0$	1.709	1.909	1.510	1.555	1.237	1.339	2.152	2.418
		(0.367)	(0.505)	(0.169)	(0.276)	(0.425)	(0.600)	(0.583)	(0.859)
	$0, (0, 10)^{*4}, 0$	1.459	1.627	1.382	1.426	1.097	1.180	1.869	2.103
		(0.457)	(0.631)	(0.266)	(0.422)	(0.578)	(0.807)	(0.573)	(0.847)
	4^{*10}	1.428	1.591	1.359	1.402	1.078	1.159	1.835	2.064
		(0.486)	(0.664)	(0.287)	(0.453)	(0.603)	(0.840)	(0.711)	(1.004)
	$40, 0^{*9}$	1.385	1.544	1.338	1.386	1.059	1.134	1.785	2.010
		(0.516)	(0.679)	(0.308)	(0.471)	(0.627)	(0.869)	(0.705)	(1.025)
20	$0^{*19}, 30$	1.798	2.022	1.716	1.871	1.449	1.586	2.097	2.392
		(0.252)	(0.363)	(0.156)	(0.199)	(0.232)	(0.330)	(0.367)	(0.530)
	$0^{*5}, 3^{*10}, 0^{*5}$	1.805	2.018	1.720	1.868	1.452	1.584	2.105	2.386
		(0.253)	(0.364)	(0.155)	(0.202)	(0.239)	(0.332)	(0.472)	(0.726)
	$0^{*5}, 3^{*10}, 0^{*5}$	1.419	1.595	1.414	1.544	1.184	1.285	1.671	1.904
		(0.382)	(0.507)	(0.295)	(0.386)	(0.469)	(0.642)	(0.377)	(0.556)
	$(1, 2)^{*10}$	1.491	1.659	1.471	1.594	1.231	1.329	1.752	1.974
		(0.333)	(0.456)	(0.252)	(0.342)	(0.415)	(0.582)	(0.388)	(0.541)
	$30, 0^{*19}$	1.355	1.512	1.364	1.482	1.141	1.229	1.601	1.808
		(0.427)	(0.548)	(0.300)	(0.432)	(0.518)	(0.708)	(0.481)	(0.729)
30	$0^{*29}, 20$	1.824	2.044	1.774	1.953	1.547	1.692	2.051	2.330
		(0.204)	(0.293)	(0.149)	(0.196)	(0.165)	(0.238)	(0.269)	(0.397)
	$0^{*5}, 1^{*20}, 0^{*5}$	1.824	2.043	1.774	1.953	1.546	1.692	2.051	2.330
		(0.204)	(0.297)	(0.149)	(0.199)	(0.166)	(0.241)	(0.271)	(0.386)
	$0^{*5}, (0, 2)^{*10}, 0^{*5}$	1.510	1.683	1.501	1.647	1.309	1.419	1.702	1.921
		(0.267)	(0.354)	(0.226)	(0.298)	(0.328)	(0.453)	(0.356)	(0.547)
	$(0, 1, 1)^{*10}$	1.575	1.763	1.558	1.715	1.358	1.479	1.774	2.012
		(0.240)	(0.319)	(0.199)	(0.259)	(0.286)	(0.394)	(0.272)	(0.396)
	$20, 0^{*29}$	1.495	1.666	1.488	1.632	1.297	1.407	1.686	1.902
		(0.272)	(0.370)	(0.230)	(0.311)	(0.336)	(0.470)	(0.357)	(0.555)
40	$0^{*39}, 10$	1.830	2.055	1.796	1.991	1.602	1.758	2.013	2.289
		(0.173)	(0.246)	(0.137)	(0.185)	(0.133)	(0.190)	(0.214)	(0.314)
	$0^{*15}, 1^{*10}, 0^{*15}$	1.835	2.052	1.800	1.989	1.606	1.757	2.018	2.286
		(0.176)	(0.249)	(0.139)	(0.186)	(0.134)	(0.191)	(0.218)	(0.439)
	$0^{*10}, (0, 1)^{*10}, 0^{*10}$	1.647	1.845	1.630	1.805	1.456	1.593	1.810	2.053
		(0.182)	(0.249)	(0.157)	(0.209)	(0.204)	(0.284)	(0.282)	(0.326)
	$(0, 0, 0, 1)^{*10}$	1.705	1.909	1.682	1.862	1.502	1.644	1.874	2.126
		(0.174)	(0.250)	(0.146)	(0.199)	(0.178)	(0.252)	(0.231)	(0.358)
	$10, 0^{*39}$	1.641	1.823	1.625	1.786	1.452	1.577	1.805	2.028
		(0.183)	(0.251)	(0.157)	(0.214)	(0.206)	(0.294)	(0.288)	(0.441)
50	0^{*50}	1.826	2.056	1.800	2.007	1.633	1.800	1.977	2.254
		(0.145)	(0.218)	(0.122)	(0.172)	(0.113)	(0.162)	(0.204)	(0.304)

Table 3. Average estimate values of relative risk rate and their MSEs (in Bracket) for *n*=30 at C_1=-2.5 and C_2= 3.

m	Scheme	$\hat{\pi}_1^*$	$\hat{\pi}_2^*$	$\tilde{\pi}_{1S}^*$	$\tilde{\pi}_{2S}^*$	$\tilde{\pi}_{1LC_1}^*$	$\tilde{\pi}_{2LC_1}^*$	$\tilde{\pi}_{1LC_2}^*$	$\tilde{\pi}_{2LC_2}^*$
10	$0^{*9}, 20$	0.525	0.475	0.508	0.492	0.484	0.467	0.529	0.513
		(0.016)	(0.016)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
	$0^{*3}, 4^{*5}, 0^{*2}$	0.524	0.476	0.507	0.493	0.483	0.468	0.528	0.513
		(0.016)	(0.016)	(0.008)	(0.009)	(0.008)	(0.008)	(0.008)	(0.008)
	$0, (0,\ 5)^{*4}, 0$	0.525	0.475	0.509	0.491	0.486	0.468	0.528	0.511
		(0.019)	(0.019)	(0.009)	(0.008)	(0.009)	(0.008)	(0.009)	(0.008)
	2^{*10}	0.523	0.477	0.508	0.492	0.485	0.469	0.528	0.511
		(0.020)	(0.020)	(0.010)	(0.010)	(0.009)	(0.008)	(0.009)	(0.009)
	$20, 0^{*9}$	0.524	0.476	0.508	0.492	0.486	0.468	0.528	0.511
		(0.021)	(0.021)	(0.010)	(0.009)	(0.009)	(0.009)	(0.010)	(0.009)
15	$0^{*14}, 15$	0.526	0.474	0.515	0.485	0.495	0.465	0.532	0.502
		(0.013)	(0.013)	(0.007)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	$0^{*5}, 3^{*5}, 0^{*5}$	0.526	0.474	0.515	0.485	0.495	0.464	0.532	0.502
		(0.013)	(0.013)	(0.008)	(0.008)	(0.007)	(0.007)	(0.009)	(0.009)
	$0^{*3}, (0,\ 3)^{*5}, 0^{*2}$	0.525	0.475	0.515	0.485	0.496	0.465	0.532	0.501
		(0.015)	(0.015)	(0.008)	(0.008)	(0.008)	(0.008)	(0.007)	(0.007)
	1^{*15}	0.525	0.475	0.515	0.485	0.495	0.465	0.532	0.502
		(0.015)	(0.015)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
	$15, 0^{*14}$	0.523	0.477	0.514	0.486	0.495	0.466	0.531	0.502
		(0.016)	(0.016)	(0.009)	(0.009)	(0.009)	(0.008)	(0.009)	(0.009)
20	$0^{*19}, 10$	0.525	0.475	0.517	0.483	0.501	0.467	0.531	0.497
		(0.011)	(0.011)	(0.007)	(0.007)	(0.007)	(0.008)	(0.008)	(0.008)
	$0^{*5}, 1^{*10}, 0^{*5}$	0.526	0.474	0.518	0.483	0.502	0.466	0.531	0.496
		(0.011)	(0.011)	(0.007)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	$0^{*5}, (0,\ 2)^{*5}, 0^{*5}$	0.526	0.474	0.518	0.482	0.502	0.466	0.531	0.496
		(0.013)	(0.013)	(0.008)	(0.008)	(0.008)	(0.007)	(0.007)	(0.007)
	$(0,1)^{*10}$	0.526	0.474	0.518	0.482	0.503	0.466	0.532	0.495
		(0.013)	(0.013)	(0.008)	(0.008)	(0.007)	(0.008)	(0.007)	(0.008)
	$10, 0^{*19}$	0.526	0.474	0.518	0.482	0.503	0.466	0.532	0.495
		(0.013)	(0.013)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
25	$0^{*24}, 5$	0.527	0.474	0.520	0.480	0.507	0.466	0.531	0.491
		(0.010)	(0.010)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$0^{*10}, 1^{*5}, 0^{*10}$	0.526	0.474	0.520	0.480	0.507	0.466	0.531	0.491
		(0.010)	(0.010)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$0^{*5}, (0,0,1)^{*5}, 0^{*5}$	0.525	0.475	0.519	0.481	0.506	0.467	0.531	0.492
		(0.010)	(0.010)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$(0,0,0,0,1)^{*5}$	0.525	0.475	0.519	0.481	0.506	0.467	0.531	0.492
		(0.010)	(0.010)	(0.006)	(0.006)	(0.007)	(0.006)	(0.006)	(0.007)
	$5, 0^{*24}$	0.525	0.475	0.519	0.481	0.506	0.468	0.530	0.492
		(0.010)	(0.010)	(0.006)	(0.006)	(0.007)	(0.006)	(0.006)	(0.007)
30	0^{*30}	0.526	0.474	0.521	0.480	0.509	0.467	0.530	0.489
		(0.009)	(0.009)	(0.006)	(0.006)	(0.006)	(0.005)	(0.006)	(0.006)

Table 4. Average estimate values of relative risk rate and their MSEs (in Bracket) for *n*=50 at *C_1=-2.5* and *C_2= 3.*

m	Scheme	$\hat{\pi}_1^*$	$\hat{\pi}_2^*$	$\tilde{\pi}_{1S}^*$	$\tilde{\pi}_{2S}^*$	$\tilde{\pi}_{1LC_1}^*$	$\tilde{\pi}_{2LC_1}^*$	$\tilde{\pi}_{1LC_2}^*$	$\tilde{\pi}_{2LC_2}^*$
10	$0^{*9}, 40$	0.527	0.473	0.508	0.492	0.484	0.466	0.529	0.512
		(0.016)	(0.016)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
	$0, 5^{*8}, 0$	0.525	0.475	0.506	0.493	0.484	0.467	0.529	0.5123
		(0.016)	(0.016)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
	$0, (0, 10)^{*4}, 0$	0.524	0.476	0.508	0.492	0.486	0.469	0.528	0.511
		(0.020)	(0.020)	(0.007)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	4^{*10}	0.522	0.478	0.508	0.492	0.485	0.466	0.527	0.512
		(0.020)	(0.020)	(0.010)	(0.009)	(0.009)	(0.009)	(0.008)	(0.008)
	$40, 0^{*9}$	0.523	0.477	0.508	0.492	0.486	0.469	0.528	0.511
		(0.021)	(0.021)	(0.010)	(0.009)	(0.009)	(0.009)	(0.010)	(0.009)
20	$0^{*19}, 30$	0.527	0.473	0.518	0.482	0.502	0.465	0.532	0.495
		(0.011)	(0.011)	(0.007)	(0.007)	(0.007)	(0.008)	(0.008)	(0.008)
	$0^{*5}, 3^{*10}, 0^{*5}$	0.526	0.474	0.518	0.482	0.502	0.466	0.532	0.496
		(0.011)	(0.011)	(0.006)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	$0^{*5}, 3^{*10}, 0^{*5}$	0.525	0.475	0.517	0.483	0.502	0.466	0.531	0.496
		(0.014)	(0.014)	(0.007)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	$(1, 2)^{*10}$	0.5250	0.475	0.518	0.482	0.502	0.466	0.531	0.496
		(0.014)	(0.014)	(0.007)	(0.007)	(0.008)	(0.007)	(0.007)	(0.008)
	$30, 0^{*19}$	0.527	0.473	0.519	0.481	0.503	0.465	0.532	0.494
		(0.015)	(0.015)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)
30	$0^{*29}, 20$	0.526	0.474	0.521	0.480	0.509	0.468	0.530	0.489
		(0.009)	(0.009)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$0^{*5}, 1^{*20}, 0^{*5}$	0.52	0.474	0.521	0.479	0.510	0.467	0.531	0.488
		(0.009)	(0.009)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$0^{*5}, (0, 2)^{*10}, 0^{*5}$	0.527	0.474	0.521	0.479	0.510	0.467	0.531	0.488
		(0.010)	(0.010)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
	$(0, 1, 1)^{*10}$	0.526	0.474	0.521	0.479	0.510	0.468	0.531	0.488
		(0.011)	(0.011)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
	$20, 0^{*29}$	0.525	0.475	0.521	0.480	0.509	0.468	0.530	0.490
		(0.010)	(0.010)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)	(0.007)
40	$0^{*39}, 10$	0.526	0.474	0.522	0.478	0.514	0.469	0.5298	0.485
		(0.007)	(0.007)	(0.005)	(0.005)	(0.006)	(0.005)	(0.005)	(0.005)
	$0^{*15}, 1^{*10}, 0^{*15}$	0.527	0.474	0.523	0.478	0.514	0.469	0.530	0.485
		(0.007)	(0.007)	(0.005)	(0.005)	(0.006)	(0.005)	(0.005)	(0.006)
	$0^{*10}, (0, 1)^{*10}, 0^{*10}$	0.527	0.474	0.523	0.477	0.514	0.469	0.530	0.485
		(0.008)	(0.008)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$(0, 0, 0, 1)^{*10}$	0.526	0.474	0.523	0.478	0.514	0.469	0.530	0.485
		(0.008)	(0.008)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
	$10, 0^{*39}$	0.526	0.474	0.522	0.478	0.513	0.469	0.530	0.485
		(0.008)	(0.008)	(0.008)	(0.008)	(0.007)	(0.007)	(0.007)	(0.008)
50	0^{*50}	0.526	0.474	0.523	0.477	0.516	0.470	0.529	0.483
		(0.005)	(0.0056)	(0.004)	(0.004)	(0.005)	(0.005)	(0.004)	(0.004)

Table 5: Average estimate values of component reliabilities and their MSEs (in Bracket) for n=30 at t_o=*1.2*, C_1=*-2.5* and C_2= *3*.

m	Scheme	$\hat{R}_1$	$\hat{R}_2$	$\tilde{R}_{1S}$	$\tilde{R}_{2S}$	$\tilde{R}_{1LC_1}$	$\tilde{R}_{2LC_1}$	$\tilde{R}_{1LC_2}$	$\tilde{R}_{2LC_2}$
10	$0^{*9}, 20$	0.467	0.502	0.399	0.412	0.378	0.395	0.425	0.433
		(0.019)	(0.020)	(0.017)	(0.023)	(0.024)	(0.030)	(0.012)	(0.017)
	$0^{*3}, 4^{*5}, 0^{*2}$	0.469	0.502	0.400	0.412	0.379	0.395	0.426	0.433
		(0.019)	(0.020)	(0.017)	(0.023)	(0.024)	(0.030)	(0.012)	(0.018)
	$0, (0,\ 5)^{*4}, 0$	0.410	0.446	0.369	0.381	0.343	0.359	0.396	0.404
		(0.031)	(0.033)	(0.026)	(0.032)	(0.034)	(0.042)	(0.019)	(0.027)
	2^{*10}	0.415	0.448	0.371	0.382	0.346	0.361	0.398	0.406
		(0.030)	(0.032)	(0.025)	(0.032)	(0.033)	(0.041)	(0.019)	(0.026)
	$20, 0^{*9}$	0.393	0.427	0.360	0.372	0.334	0.350	0.388	0.396
		(0.036)	(0.039)	(0.028)	(0.036)	(0.037)	(0.045)	(0.022)	(0.030)
15	$0^{*14}, 15$	0.487	0.522	0.440	0.462	0.422	0.445	0.460	0.481
		(0.013)	(0.014)	(0.011)	(0.014)	(0.015)	(0.018)	(0.009)	(0.010)
	$0^{*5}, 3^{*5}, 0^{*5}$	0.486	0.522	0.440	0.462	0.422	0.446	0.460	0.481
		(0.013)	(0.014)	(0.011)	(0.014)	(0.015)	(0.018)	(0.009)	(0.010)
	$0^{*3}, (0,\ 3)^{*5}, 0^{*2}$	0.430	0.467	0.399	0.421	0.378	0.401	0.420	0.442
		(0.022)	(0.023)	(0.020)	(0.023)	(0.026)	(0.030)	(0.016)	(0.018)
	1^{*15}	0.430	0.465	0.399	0.420	0.378	0.400	0.420	0.442
		(0.023)	(0.023)	(0.020)	(0.023)	(0.026)	(0.030v	(0.016)	(0.018)
	$15, 0^{*14}$	0.402	0.436	0.380	0.400	0.358	0.378	0.401	0.422
		(0.029)	(0.031)	(0.025)	(0.029)	(0.032)	(0.037)	(0.020)	(0.024)
20	$0^{*19}, 10$	0.496	0.530	0.461	0.485	0.445	0.470	0.476	0.500
		(0.010)	(0.011)	(0.009)	(0.010)	(0.011)	(0.013)	(0.007)	(0.008)
	$0^{*5}, 1^{*10}, 0^{*5}$	0.495	0.530	0.459	0.484	0.444	0.470	0.475	0.500
		(0.010)	(0.011)	(0.009)	(0.010)	(0.011)	(0.013)	(0.007)	(0.008)
	$0^{*5}, (0,\ 2)^{*5}, 0^{*5}$	0.459	0.495	0.431	0.456	0.414	0.440	0.447	0.473
		(0.015)	(0.016)	(0.014)	(0.015)	(0.017)	(0.019)	(0.011)	(0.012)
	$(0, 1)^{*10}$	0.441	0.479	0.417	0.443	0.399	0.426	0.434	0.461
		(0.018)	(0.018)	(0.016)	(0.018)	(0.020)	(0.023)	(0.013)	(0.015)
	$10, 0^{*19}$	0.437	0.474	0.413	0.440	0.396	0.422	0.430	0.457
		(0.019)	(0.019)	(0.017)	(0.019)	(0.021)	(0.024)	(0.014)	(0.016)
25	$0^{*24}, 5$	0.500	0.536	0.472	0.500	0.459	0.487	0.484	0.513
		(0.009)	(0.009)	(0.007)	(0.008)	(0.009)	(0.010)	(0.006)	(0.007)
	$0^{*10}, 1^{*5}, 0^{*10}$	0.500	0.536	0.472	0.500	0.460	0.488	0.485	0.513
		(0.009)	(0.009)	(0.007)	(0.008)	(0.009)	(0.010)	(0.006)	(0.007)
	$0^{*5}, (0, 0, 1)^{*5}, 0^{*5}$	0.478	0.512	0.453	0.479	0.439	0.466	0.466	0.493
		(0.011)	(0.012)	(0.010)	(0.011)	(0.012)	(0.014)	(0.009)	(0.010)
	$(0, 0, 0, 0, 1)^{*5}$	0.489	0.524	0.462	0.490	0.449	0.477	0.475	0.503
		(0.010)	(0.010)	(0.009)	(0.010)	(0.010)	(0.012)	(0.007)	(0.008)
	$5, 0^{*24}$	0.478	0.513	0.453	0.480	0.440	0.467	0.466	0.494
		(0.011)	(0.012)	(0.010)	(0.011)	(0.012)	(0.014)	(0.009)	(0.010)
30	0^{*30}	0.504	0.540	0.481	0.509	0.469	0.498	0.491	0.520
		(0.007)	(0.008)	(0.006)	(0.007)	(0.007)	(0.008)	(0.006)	(0.006)

Table 6. Average estimate values of component reliabilities and their MSEs (in Bracket) for n=50 at t_o=*1.2*, C_1=*-2.5* and C_2= *3*.

m	Scheme	$\hat{R}_1$	$\hat{R}_2$	$\tilde{R}_{1S}$	$\tilde{R}_{2S}$	$\tilde{R}_{1LC_1}$	$\tilde{R}_{2LC_1}$	$\tilde{R}_{1LC_2}$	$\tilde{R}_{2LC_2}$
10	$0^{*9}, 40$	0.467	0.504	0.400	0.413	0.378	0.397	0.426	0.434
		(0.019)	(0.020)	(0.017)	(0.023)	(0.024)	(0.029)	(0.012)	(0.017)
	$0, 5^{*8}, 0$	0.467	0.501	0.399	0.411	0.378	0.395	0.425	0.433
		(0.019)	(0.020)	(0.017)	(0.023)	(0.024)	(0.030)	(0.012)	(0.017)
	$0, (0, 10)^{*4}, 0$	0.400	0.434	0.363	0.375	0.337	0.352	0.390	0.398
		(0.034)	(0.036)	(0.027)	(0.035)	(0.036)	(0.044)	(0.021)	(0.029)
	4^{*10}	0.408	0.440	0.367	0.379	0.342	0.356	0.395	0.402
		(0.032)	(0.035)	(0.026)	(0.033)	(0.035)	(0.043)	(0.020)	(0.028)
	$40, 0^{*9}$	0.386	0.419	0.356	0.368	0.330	0.344	0.384	0.392
		(0.038)	(0.041)	(0.029)	(0.037)	(0.039)	(0.047)	(0.023)	(0.032)
20	$0^{*19}, 30$	0.495	0.531	0.459	0.485	0.444	0.471	0.475	(0.501
		(0.010)	(0.011)	(0.009)	(0.010)	(0.011)	(0.013)	(0.007)	(0.008)
	$0^{*5}, 3^{*10}, 0^{*5}$	0.495	0.531	0.460	0.485	0.445	0.471	0.476	0.501
		(0.011)	(0.011)	(0.009)	(0.010)	(0.011)	(0.013)	(0.007)	(0.008)
	$0^{*5}, 3^{*10}, 0^{*5}$	0.425	0.461	0.404	0.430	0.386	0.412	0.422	0.448
		(0.021)	(0.022)	(0.019)	(0.022)	(0.023)	(0.027)	(0.016)	(0.018)
	$(1, 2)^{*10}$	0.406	0.442	0.390	0.416	0.372	0.397	0.407	0.434
		(0.026)	(0.027)	(0.023)	(0.025)	(0.028)	(0.031)	(0.019)	(0.021)
	$30, 0^{*19}$	0.389	0.429	0.378	0.405	0.359	0.386	0.395	0.424
		(0.030)	(0.030)	(0.026)	(0.028)	(0.032)	(0.035)	(0.022)	(0.024)
30	$0^{*29}, 20$	0.504	0.540	0.481	0.510	0.470	0.499	0.491	0.521
		(0.007)	(0.008)	(0.006)	(0.007)	(0.007)	(0.008)	(0.006)	(0.006)
	$0^{*5}, 1^{*20}, 0^{*5}$	0.505	0.540	0.481	0.510	0.470	0.499	0.492	0.520
		(0.007)	(0.008)	(0.006)	(0.007)	(0.007)	(0.008)	(0.006)	(0.006)
	$0^{*5},, (0, 2)^{*10}, 0^{*5}$	0.451	0.488	0.434	0.464	0.421	0.452	0.446	0.477
		(0.013)	(0.013)	(0.013)	(0.014)	(0.015)	(0.016)	(0.011)	(0.012)
	$(0, 1, 1)^{*10}$	0.434	0.471	0.419	0.450	0.407	0.436	0.431	0.462
		(0.016)	(0.016)	(0.016)	(0.017)	(0.018)	(0.020)	(0.014)	(0.014)
	$20, 0^{*29}$	0.430	0.467	0.417	0.446	0.404	0.433	0.429	0.459
		(0.017)	(0.017)	(0.016)	(0.018)	(0.019)	(0.021)	(0.014)	(0.015)
40	$0^{*39}, 10$	0.508	0.543	0.490	0.521	0.481	0.512	0.498	0.529
		(0.006)	(0.006)	(0.005)	(0.006)	(0.006)	(0.006)	(0.005)	(0.005)
	$0^{*15}, 1^{*10}, 0^{*15}$	0.508	0.544	0.490	0.521	0.482	0.513	0.499	0.530
		(0.006)	(0.006)	(0.005)	(0.006)	(0.006)	(0.006)	(0.005)	(0.005)
	$0^{*10}, (0, 1)^{*10}, 0^{*10}$	0.482	0.519	0.467	0.499	0.458	0.489	0.475	0.507
		(0.008)	(0.008)	(0.007)	(0.008)	(0.008)	(0.009)	(0.007)	(0.007)
	$(0, 0, 0, 1)^{*10}$	0.470	0.507	0.456	0.487	0.446	0.478	0.464	0.496
		(0.009)	(0.009)	(0.009)	(0.009)	(0.010)	(0.011)	(0.008)	(0.008)
	$10, 0^{*39}$	0.468	0.50	0.454	0.486	0.445	0.476	0.463	0.495
		(0.009)	(0.009)	(0.009)	(0.010)	(0.010)	(0.011)	(0.008)	(0.009)
	0^{*50}	0.510	0.545	0.496	0.527	0.489	0.520	0.503	0.534
50		(0.005)	(0.005)	(0.004)	(0.005)	(0.005)	(0.005)	(0.004)	(0.004)