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# On Certain Topological Structures of Summable Paranormed Sequence Space Defined in Two - Normed Space

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## ABSTRACT

The aim of this paper is to introduce and study a new class $\ell\,(S, \overline{\xi}, \overline{u}\,)$ of sequences with values in *2*- Banach space as a generalization of the familiar space of summable sequences $\ell$. We explore some of the preliminary results that characterize the topological linear structure of the class $\ell\,(S, \overline{\xi}, \overline{u}\,)$ when topologized it with suitable natural paranorm.

**Keywords:** paranormed space, *2*- normed space, sequence space, solid space.

## INTRODUCTION

So far, a bulk number of works have been done on various types of paranormed spaces. The concept of paranorm is closely related to linear metric space and its studies on sequence spaces were initiated by Maddox (1969) and many others.

Before proceeding with the main results,we begin with recalling some of the notations and basic definitions that are used in this paper.

**Definition 1:** A paranormed space $(S, P\,)$ is a linear space $S$ with zero element $\theta$ together with a function $P : S \to \boldsymbol{R}^+$ (called a paranorm on $S$) which satisfies the following axioms:

(i) $P\,(\theta) = 0$;

(ii) $P\,(s) = P\,(-s)$, for all $s \in S$;

(iii) $P\,(s_1 + s_2\,) \leq P\,(s_1) + P\,(s_2\,)$ , for all $s_1$ , $s_2 \in S$; and

(iv) Scalar multiplication is continuous i.e., if $< \gamma_n >$ is a sequence of scalars with $\gamma_n \to \gamma$ as $n \to \infty$ and $< s_n >$ is a sequence of vectors with $P\,(s_n - s\,) \to 0$ as $n \to \infty$, then $P\,(\gamma_n\, s_n - \gamma s) \to 0$ as $n \to \infty$. Note that the continuity of scalar multiplication is equivalent to

(i) if $P\,(s_n) \to 0$ and $\gamma_n \to \gamma$ as $n \to \infty$, then $P\,(\gamma_n\, s_n) \to 0$ as $n \to \infty$; and

(ii) if $\gamma_n \to 0$ as $n \to \infty$ and $s$ be any element in $S$, then $P\,(\gamma_n\, s) \to 0$, see Wilansky (1978).

A paranorm is called total if $P\,(s) = 0 \Rightarrow s = \theta$, see Wilansky (1978).

The studies of paranorms on sequence spaces were initiated by Maddox (1969) and many others. Parasar and Choudhary (1994), Bhardwaj and Bala 2007), Khan (2008), Basariv and Altundag (2009), Tiwari and Srivastava (2010), Pahari (2011, 2013, 2013, 2014 & 2014) , and many others further studied various types of paranormed sequence spaces and function spaces.

**Definition 2:** Let $S$ be a linear space of dimension > 1 over $\boldsymbol{K}$, the field of real or complex numbers. A 2 - norm on $S$ is a real valued function $\|.\,,\,.\|$ on $S \times S$ satisfying the following conditions:

(i) $\|\, s, t\, \| \geq 0$ and $\|\, s, t\, \| = 0$ if and only if $s$ and $t$ are linearly dependent;

(ii) $\|\, s, t\, \| = \|\, t\,, s\, \|$, for all $s, t \in S$;

(iii) $\|\, \alpha s, t\, \| = |\alpha\,|\, \|\, s\,,\, t\, \|$, where $\alpha \in \boldsymbol{K}$ and $s, t \in S$;and

(iv) $\|\, s_1 + s_2, t\, \| \leq \|\, s_1, t\, \| + \|\, s_2, t\, \|$ ,for all $s_1$ , $s_2$ and $t \in S$.

The pair $(S, \|.\,,\,.\|)$ is called a 2–normed space. Thus the notion of 2–normed space is just a two- dimensional analogue of a normed space.

The concept of 2–normed space was initially introduced by S. GÄahler (1963) as an interesting linear generalization of a normed linear space, which was subsequently studied by Iseki (1976), White and Cho (1984), Freese *et al.* (1992), Freese and Cho (2001) and many others. Recently a lot of activities have been started by many researchers to study this concept in different directions, for instances, Gunawan and Mashadi (2001) , Açikgöz (2007), Savas (2010) , Srivastava and Pahari (2011& 2013), and others.

Recall that $(S, \|.\,,.\|)$ is a 2-Banach space if every Cauchy sequence $< s_n >$ in $S$ is convergent to some $s_0$ in $S$. Geometrically, a 2-norm function represents the area of the usual parallelogram spanned by the two associated vectors. As an example, consider $S = \boldsymbol{R}^2$, being equipped with $\| \bar{s}, \bar{t} \| = | s_1 t_2 - s_2 t_1 |$, where $\bar{s} = (s_1, s_2)$ and $\bar{t} = (t_1, t_2)$. Then $(S, \|.\,,.\|)$ forms a 2–normed space and $\| \bar{s}, \bar{t} \|$ represents the area of the parallelogram spanned by the two associated vectors $\bar{s}$ and $\bar{t}$.

**Definition 3:** Let $( S, \|.\,,.\| )$ be the 2- Normed space over the field $\boldsymbol{C}$ of complex numbers and $\bar{\theta} = (\theta, \theta, \theta ,... )$ denotes the zero element of $S$. Let $\omega(S)$ denotes the linear space of all sequences $\bar{s} = < s_k >$ with $s_k \in S$ , $k \geq 1$ with usual coordinate wise operations i.e., for each

$\bar{s} = < s_k >$ , $\bar{w} = < w_k > \in \omega(S)$ and $\gamma \in \boldsymbol{C}$,

$\bar{s} + \bar{w} = < s_k + w_k >$ and $\gamma \bar{s} = < \gamma s_k >$ .

We shall denote $\omega(\boldsymbol{C})$ by $\omega$. Any linear subspace of $\omega$ is then called a *sequence space*.

Further, if $\bar{\gamma} = < \gamma_k > \in \omega$ and $\bar{s} \in \omega(S)$ we shall write $\bar{\gamma}\bar{s} = < \gamma_k s_k >$ .

**Definition 4:** A sequence space $S$ is said to be *solid* if $\bar{s} = < s_k > \in S$ and $\bar{\gamma} = < \gamma_k >$ a sequence of scalars with $|\gamma_k| \leq 1$, for all $k \geq 1$, then $\bar{\gamma}\bar{s} = < \gamma_k s_k > \in S$.

**Definition 5:** A sequence $\bar{s} = < s_n >$ in a linear 2–normed space $S$ is *convergent* if there is an $s_0 \in S$ such that $\lim_{n\to\infty} \| s_n - s, t \| = 0$, for each $t \in S$ .It is said to be a *Cauchy* if there are $t$ and $w$ in $S$ such that $t$ and $w$ are linearly independent and $\lim_{m,n\to\infty} \|s_m - s_n, t \| = 0$ and $\lim_{m,n\to\infty} \|s_m - s_n, w \| = 0$.

The notion of convergence was introduced by White and Cho (1984).A linear 2–normed space $(S, \|.\,,.\|)$ is called *2 Banach* space if every Cauchy sequence $< s_n >$ in $S$ is convergent to some $s \in S$.

## The Class $\ell$ $(S, \bar{\xi}, \bar{u})$ of 2-Normed Space Valued Sequences

Let $\bar{u} = < u_k >$ and $\bar{v} = < v_k >$ be any sequences of strictly positive real numbers and $\bar{\xi} = < \xi_k >$ and $\bar{\mu} = < \mu_k >$ be the sequences of non zero complex numbers.

We now introduce the following classes of 2-normed space $S$–valued vector sequences

$\ell( S, \quad , \bar{u} ) = \{ \bar{s} = < s_k > \in \omega(S)$ satisfying $\sum_{k=1}^{\infty} \| \xi_k s_k, t \|^{u_k} < \infty$ ,for each $t \in S \}$.

In fact, this class is a generalization of the familiar sequence spaces, studied in Pahari (2011, 2013 & 2014), Srivastava and Pahari (2011, 2011 & 2013) , using 2-norm

## RESULTS

In this section, we shall investigate some results that characterize the linear topological structure of the class $\ell$ $(S, \bar{\xi}, \bar{u})$ of 2-normed space $S$- valued sequences by endowing it with suitable natural paranorm. Throughout the work, we denote $z_k = |\xi_k \mu_k^{-1}|^{u_k}$ , $\sup u_k = M$ and for scalar $\alpha$, $A[\alpha] = \max(1, |\alpha|)$. But when the sequences $< u_k >$ and $< v_k >$ occur, then to distinguish $M$ we use the notations $M(u)$ and $M(v)$ respectively.

**Theorem 1:** The space $\ell(S, \bar{\xi}, \bar{u})$ forms a solid.

Proof.

Let $\bar{s} = < s_k > \in \ell(S, \bar{\xi}, \bar{u})$ . So that for each $t \in S$,

$$\sum_{k=1}^{\infty} \|\xi_k s_k , t \|^{u_k} < \infty.$$

Let $< \gamma_k >$ be a sequence of scalars satisfying $|\gamma_k| \leq 1$ for all $k \geq 1$.Then we have

$$\sum_{k=1}^{\infty} \|\xi_k \gamma_k s_k, t \|^{u_k} = \sum_{k=1}^{\infty} |\gamma_k|^{u_k} \| \xi_k s_k, t \|^{u_k}$$

$$\leq \sum_{k=1}^{\infty} \|\xi_k s_k , t \|^{u_k} < \infty,$$

for each $t \in S$. This shows that $< \gamma_k s_k > \in \ell(S, \bar{\xi}, \bar{u})$ and hence $\ell(S, \bar{\xi}, \bar{u})$ is normal.

**Theorem 2:** For any $\bar{u} = \langle u_k \rangle$, $\ell(S, \bar{\xi}, \bar{u}) \subset \ell(S, \bar{\mu}, \bar{u})$ if *lim inf*$_k$ $z_k > 0$.

Proof.

Assume that lim inf$_k$ $z_k > 0$ and $\bar{s} = \langle s_k \rangle \in \ell(S, \bar{\xi}, \bar{u})$. Then there exist $m > 0$ and a positive integer $K$ such that $m\,|\mu_k|^{u_k} < |\xi_k|^{u_k}$ for all $k \geq K$ and for each $t \in S$,

$\sum_{k=1}^{K} \|\xi_k s_k, t\|^{u_k} < \infty$. Thus for each $t \in S$, we have

$$\sum_{k=1}^{K} \|\mu_k s_k, t\|^{u_k} \leq \sum_{k=1}^{K} \frac{|\xi_k|^{u_k}}{m} \|s_k, t\|^{u_k}$$

$$= \frac{1}{m} \sum_{k=1}^{K} \|\xi_k s_k, t\|^{u_k} < \infty .$$

This clearly implies that $\bar{s} \in \ell(S, \bar{\mu}, \bar{u})$ and hence

$\ell(S, \bar{\xi}, \bar{u}) \subset \ell(S, \bar{\mu}, \bar{u})$. This completes the proof.

**Theorem 3:** For any $\bar{\xi} = \langle \xi_k \rangle$, *if* $u_k \leq v_k$ for all but finitely many values of $k$, then $\ell(S, \bar{\xi}, \bar{u}) \subset \ell(S, \bar{\xi}, \bar{v})$.

Proof.

Suppose $0 < u_k \leq v_k < \infty$ for all but finitely many values of $k$. Let $\bar{s} = \langle s_k \rangle \in \ell(S, \bar{\xi}, \bar{u})$. Then we have $\sum_{k=1}^{\infty} \|\xi_k s_k, t\|^{u_k} < \infty$, for each $t \in S$.

This shows that there exists $K \geq 1$ such that $\|\xi_k\ s_k, t\| < 1$ for all $k \geq K$ and for each $t \in S$. Thus $\|\xi_k s_k, t\|^{v_k} \leq \|\xi_k s_k, t\|^{u_k}$ for all $k \geq K$ and for each $t \in S$ and consequently

$$\sum_{k=1}^{K} \|\xi_k s_k, t\|^{v_k} \leq \sum_{k=1}^{K} \|\xi_k s_k, t\|^{u_k} < \infty \text{, for each } t \in S$$

and hence $\bar{s} \in \ell(S, \bar{\xi}, \bar{v})$. This completes the proof of the theorem. The following result is an immediate consequence of Theorems 2 and 3.

**Theorem 4:** If (i) *lim inf*$_k$ $z_k > 0$; and

(ii) $u_k \leq v_k$, for all but finitely many values of $k$, then $\ell(S, \bar{\xi}, \bar{u}) \subset \ell(S, \bar{\mu}, \bar{v})$.

In the following example, we conclude that $\ell(S, \bar{\xi}, \bar{u})$ may strictly be contained in $\ell(S, \bar{\mu}, \bar{v})$ inspite of the satisfaction of both conditions of Theorem 4.

**Example 5:** Let $(S, \|.\,,.\|)$ be a 2- normed space and consider a sequence $\bar{s} = \langle s_k \rangle$ defined by $s_k = \frac{1}{k^{2k}} s$, if $k = 1, 2, 3, \ldots$, where $s \in S$ and $s \neq \theta$.

Further, let $u_k = \frac{1}{k}$, if $k$ is odd integer, $u_k = \frac{1}{k^2}$, if $k$ is even integer, $v_k = \frac{1}{k}$ for all values of $k$, $\xi_k = 3^k$, $\mu_k = 2^k$ for all values of $k$.

Then, $z_k = \left|\frac{\xi_k}{\mu_k}\right|^{u_k} = \frac{3}{2}$ or $\left(\frac{3}{2}\right)^{1/k}$ according as $k$ is odd or even integers and hence lim inf$_k$ $z_k > 0$.

Further, $\frac{v_k}{u_k} = 1$, if $k$ is odd integers, $\frac{v_k}{u_k} = k$, if $k$ is even integers. Therefore $0 < u_k \leq v_k < \infty$ for all $k$.

Hence both the conditions of Theorem 4 are satisfied.

Now for each $t \in S$, we have

$$\sum_{k=1}^{\infty} \|\mu_k s_k, t\|^{v_k} = \sum_{k=1}^{\infty} \|2^k \frac{1}{k^{2k}}\ s, t\|^{1/k}$$

$$= \sum_{k=1}^{\infty} \frac{2}{k^2} \|s, t\|^{1/k}$$

$$\leq 2A\,[\,\|s, t\|\,] \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

This shows that $\bar{s} \in \ell(S, \bar{\mu}, \bar{v})$. But on the other hand, let us choose $t \in S$ such that $\|s, t\| = 1$. Then for each even integer $k$, we have

$$\|\xi_k s_k, t\|^{u_k} = \|3^k \frac{1}{k^{2k}}\ y, t\|^{1/k^2}$$

$$= \left(\frac{3}{k^2}\right)^{1/k} \|s, t\|^{1/k^2} > \frac{1}{2}.$$

This implies that $\bar{s} \notin \ell(S, \bar{\xi}, \bar{u})$ and hence the containment of $\ell(S, \bar{\xi}, \bar{u})$ in $\ell(S, \bar{\mu}, \bar{v})$ is strict.

**Theorem 6:** $\ell(S, \bar{\xi}, \bar{u})$ forms a linear space over the field of complex numbers $\boldsymbol{C}$ if $\langle u_k \rangle$ is bounded above.

Proof.

Assume that $sup_k\, u_k < \infty$ and $\overline{s} = < s_k >, \overline{w} = < w_k >$ $\in \ell(S, \overline{\xi}, \overline{u})$ .So that for each $t \in S$, we have

$$\sum_{k=1}^{\infty} \|\xi_k s_k, t\|^{u_k} < \infty \text{ and } \sum_{k=1}^{\infty} \|\xi_k w_k, t\|^{u_k} < \infty.$$

Let $0 < u_k \leq \sup_k u_k = M, T = \max(1, 2^{M-1})$ and setting $2T \max(1, |\alpha|^M) \leq 1$ and $2T \max(1, |\beta|^M) \leq 1$ and using $|a + b|^{u_k} \leq T\{|a|^{u_k} + |b|^{u_k}\}$ for all $a, b \in \mathbf{C}$.

Then we have

$$\sum_{k=1}^{\infty} \|\xi_k(\alpha s_k + \beta\, w_k), t\|^{u_k}$$

$$\leq \sum_{k=1}^{\infty} [\, T|\alpha|^{u_k} \|\xi_k s_k, t\|^{u_k} + T\, |\beta|^{u_k} \|\xi_k w_k, t\|^{u_k}\,]$$

$$\leq \sum_{k=1}^{\infty} [\, T\ A\,[|\alpha|^M]\, \|\xi_k s_k, t\|^{u_k} + T\, A\,[|\beta|^M]\, \|\xi_k w_k, t\|^{u_k}]$$

$$\leq \frac{1}{2} \sum_{k=1}^{\infty} \|\xi_k s_k, t\|^{u_k} + \frac{1}{2} \sum_{k=1}^{\infty} \|\xi_k w_k, t\|^{u_k} < \infty,$$

for each $t \in S$ and therefore $\alpha\overline{s} + \beta\overline{w} \in \ell((S, \overline{\xi}, \overline{u})$.

This implies that $\ell((S, \overline{\xi}, \overline{u})$ forms a linear space over $\boldsymbol{C}$.

**Theorem 7:** If $\ell((S, \overline{\xi}, \overline{u})$ forms a linear space over $\boldsymbol{C}$ then $< u_k >$ is bounded above.

Proof.

Suppose that $\ell(S, \overline{\xi}, \overline{u})$ forms a linear space over $\boldsymbol{C}$ but $sup_k\, u_k = \infty$. Then there exists a sequence $< k(n) >$ of positive integers satisfying $1 \leq k(n) < k(n + 1), n \geq 1$

for which

$u_{k(n)} > n$ , for each $n \geq 1$ ………………..(1)

Now, corresponding to $s_0 \in S$ and $s_0 \neq \theta$, we define the sequence $\overline{s} = < s_k >$ by

$$s_k = \begin{cases} \xi_{k(n)}^{-1}\, n^{-2/u_{k(n)}} s_0, \text{ if } k = k(n)\ ,\ n \geq 1 \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \quad \ldots(2)$$

Then for $k = k(n)$, $n \geq 1$, we have

$$\sum_{k=1}^{\infty} \|\xi_k s_k, t\|^{u_k} = \sum_{n=1}^{\infty} \|n^{-2/u_{k(n)}} s_0, t\|^{u_{k(n)}})$$

$$= \sum_{n=1}^{\infty} \frac{\|s, s_0\|^{u_{k(n)}}}{n^2}$$

$$\leq A\,[\,\|s_0, t\|^{M(u)}\,] \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

and $\|\xi_k s_k, t\|^{u_k} = 0$, for $k \neq k(n)$ , $n \geq 1$,

showing that $\overline{s} \in \ell(S, \overline{\xi}, \overline{u})$.

But on the other hand, let us choose $t_0 \in S$ such that $\|s_0, t_0\| = 1$. Then for such $t_0$ and scalar

$\alpha = 4$, for $k = k(n)$, $n \geq 1$, in view of (1) and (2), we have

$$\sum_{k=1}^{\infty} \|\alpha\, \xi_k s_k, t_0\|^{u_k} = \sum_{n=1}^{\infty} \|\xi_{k(n)}\, \alpha\, s_{k(n)}, t_0\|^{u_{k(n)}}$$

$$= \sum_{n=1}^{\infty} \|4\ n^{-2/u_{k(n)}}\ s_0, t_0\|^{u_{k(n)}}$$

$$= \sum_{n=1}^{\infty} \frac{4^{u_{k(n)}}}{n^2} \|s_0, t_0\|$$

$$\geq \sum_{n=1}^{\infty} \frac{4^n}{n^2} > 1\,.$$

This shows that $\alpha\,\overline{s} \notin \ell(S, \overline{\xi}, \overline{u})$ , a contradiction. This completes the proof.

The following result is an immediate consequence of Theorems 6 and 7.

**Theorem 8:** $\ell(S, \overline{\xi}, \overline{u})$ is a linear space over $\boldsymbol{C}$ if and only if $sup_k\, u_k < \infty$.

Let $\overline{u} = < u_k >$ such that $\sup_k u_k < \infty$ and $\overline{s} = < s_k > \in \ell(S, \overline{\xi}, \overline{u})$. We define a real valued function

$$P_{\xi,u}(\overline{s}) = \{ (\sum_{k=1}^{\infty} \|\xi_k s_k, t\|^{u_k})^{1/M}, \text{for each } t \in S \}. \ldots(3)$$

Throughout the work, $P$ will denote $P_{\xi,u}$ and $\overline{u} = < u_k >$, $\overline{v} = < v_k >$ such that $\sup_k u_k < \infty$ and $\sup_k v_k < \infty$.

We prove below that $\ell(S, \overline{\xi}, \overline{u})$ with respect to $P$ forms a paranormed space.

**Theorem 9:** $\ell(S, \overline{\xi}, \overline{u})$ forms a total paranormed -space with respect to $P$.

Proof.

Let $\alpha \in C$ and $\overline{s} = \langle s_k \rangle$, $\overline{w} = \langle w_k \rangle \in \ell(S, \overline{\xi}, \overline{u})$.Then we can easily verify that $P$ satisfy the following properties of paranorm.

(i) $P(\overline{s}) \geq 0$, and $P(\overline{s}) = 0$ if and only if $\overline{s} = \overline{\theta}$;

(ii) $P(\overline{s} + \overline{w}) \leq P(\overline{s}) + P(\overline{w})$;

(iii) $P(\alpha\overline{s}) \leq A(\alpha)\, P(\overline{s})$;

(iv) Finally for continuity of scalar multiplication, it is sufficient to show that

(a) $P(\overline{s}^{(n)}) \to 0$ and $\gamma_n \to \gamma$ imply $P(\gamma_n \overline{s}^{(n)}) \to 0$; and

(b) $\gamma_n \to 0$ implies $P(\gamma_n \overline{s}) \to 0$ for each $\overline{s} \in \ell(S, \overline{\xi}, \overline{u})$.

Now to prove (a) suppose $|\gamma_n| \leq L$ for all $n \geq 1$, then in view of (3) , we have

$$P(\gamma_n \overline{s}^{(n)}) = \{ (\sum_{k=1}^{\infty} \| \gamma_n\ \xi_k s_k, t \|^{u_k})^{1/M}, \text{ for each } t \in S \}$$

$$\leq \sup_k |\gamma_n|^{u_k/M} \{ (\sum_{k=1}^{\infty} \| \xi_k s_k, t \|^{u_k})^{1/M}, \text{ for each } t \in S \}$$

$$\leq A(L)\, P(\overline{s}^{(n)}), \text{ whence (a) follows.}$$

Next if $\overline{s} \in \ell(S, \overline{\xi}, \overline{u})$ , then for $\varepsilon > 0$ there exists an integer $K$ such that

$$\sum_{k=1}^{K} \| \xi_k s_k, t \|^{u_k} < \left(\frac{\varepsilon}{2}\right)^M, \text{ for each } t \in S.$$

Further if $\gamma_n \to 0$, we can find $N$ such that for $n \geq N$, then for each $t \in S$, we have

$$\sum_{k=1}^{K-1} |\gamma_n|^{u_k} \| \xi_k s_k, t \|^{u_k} < \left(\frac{\varepsilon}{2}\right)^M \text{ and } |\gamma_n| \leq 1.$$

Thus for each $t \in S$, $P(\gamma_n \overline{s}) \leq \left( \sum_{k=1}^{K-1} \| \gamma_n\ \xi_k s_k, t \|^{u_k} \right)^{1/M}$

$$+ \left( \sum_{k=1}^{K} \| \xi_k s_k, t \|^{u_k} \right)^{1/M} < \varepsilon,$$

for all $n \geq N$ , and hence (b) follows.

**Theorem 10:** If $S$ is a Banach space, then $(\ell(S, \overline{\xi}, \overline{u}), P)$ is complete.

Proof.

We prove the completeness of $\ell(S, \overline{\xi}, \overline{u})$ with respect to the metric $d(\overline{s}, \overline{t}) = P(\overline{s} - \overline{t})$.

Let $\langle \overline{s}^{(n)} \rangle$ be a Cauchy sequence in $\ell(S, \overline{\xi}, \overline{u})$ . Then for $0 < \varepsilon < 1$ , there exists $N$ such that for all $n, m \geq N$ and for each $t \in S$, we have $P(\overline{s}^{(n)} - \overline{s}^{(m)})$

$$= (\sum_{k=1}^{\infty} \| \xi_k s_k^{(n)} - \xi_k s_k^{(m)}, t \|^{u_k})^{1/M} < \varepsilon. \quad \text{……….. (4)}$$

and so for all $n, m \geq N$ and $k \geq 1$ and for each $t \in S$, we have

$$\| s_k^{(n)} - s_k^{(m)}, t \| < |\xi_k|^{-1} \varepsilon^{M/u_k} < |\xi_k|^{-1} \varepsilon.$$

This shows that for each $k$, $\langle s_k^{(n)} \rangle$ is a Cauchy sequence in $S$ and because of completeness of $S$, $s_k^{(n)} \to s_k \in S$ (say) for each $k$. Being a Cauchy sequence $\langle s_k^{(n)} \rangle$ is bounded, i.e. $P(s_k^{(n)}) \leq L$ for some $L > 0$ and for all $n \geq 1$. Thus for every $n$ and $r$, $(\sum_{k=1}^{r} \| \xi_k s_k^{(n)} - \xi_k, t \|^{u_k})^{1/M} \leq L$.

First taking $n \to \infty$ and then $r \to \infty$, then for each $t \in S$, $(\sum_{k=1}^{\infty} \|\xi_k s_k, t \|^{u_k})^{1/M} \leq L$ which implies that $\overline{s} = \langle s_k \rangle \in \ell(S, \overline{\xi}, \overline{u})$ .

Now for any $r$, by (4) we have

$(\sum_{k=1}^{r} \| \xi_k s_k^{(n)} - \xi_k s_k^{(m)}, t \|^{u_k})^{1/M} < \varepsilon$, for $n, m \geq N$, and so letting $m \to \infty$ first and then $r \to \infty$, we get $P(\overline{s}^{(n)} - \overline{s})$

$$= (\sum_{k=1}^{\infty} \|\xi_k s_k^{(n)} - \xi_k s_k, t \|^{u_k})^{1/M} \leq \varepsilon, \text{ for all } n \geq N \text{ and}$$

for each $t \in S$ i.e. $\overline{s}^{(n)} \to \overline{s}$ in $\ell(S, \overline{\xi}, \overline{u})$ , as $n \to \infty$.

This proves the completeness of $\ell(S, \overline{\xi}, \overline{u})$.

## CONCLUSION

In the present work, we have studied some of the conditions that typify the topological structures and containment relations of 2-normed space valued summable sequences. In fact, this result can be used for further study to explore other properties of the 2-normed space valued sequences and functions.

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