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Estimation of the Largest and the Smallest Function Values of a Feasible Solution for the Total Product Rate Variation Problem

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ABSTRACT
The problem of minimizing the total deviations between the actual and the ideal cumulative production of a variety of models of a common base product arises as a sequencing problem in mixed-model just-in-time production systems. This is called the total product rate variation problem. Several pseudo-polynomial exact algorithms and heuristics have been derived for this problem. In this paper, we estimate the largest and the smallest function values of a feasible solution for the problem when the m-th power of the total deviations between the actual and the ideal cumulative productions has to be minimized.
Key words: Bound, Product rate variation problem, Non-linear integer programming problem.

INTRODUCTION
Many companies have changed the assembly lines from paced single-model lines for mass production to mixed-model assembly lines for mass customization of a variety of models of a common base product. Just-in-time production system which requires producing only the necessary products in the necessary quantities at the necessary times usually uses mixed-model assembly lines.
Mixed-model just-in-time production systems with negligible change-over costs between the models have been used in order to respond to the customer demands for a variety of models of a common base product without holding large inventories or incurring large shortages. One of the most important problems for the effective utilization of the systems consists in sequencing different models with keeping the rate of usage of all parts used by the assembly lines as constant as possible. This problem is known as the mixed-model just-in-time sequencing problem (abbreviated as MMJITSP). The problem of minimizing the variation in the rate at which different models are produced on the line is called the product rate variation problem (abbreviated as PRVP). The latter problem is the single-level case of MMJITSP. The problem of minimizing the total deviations between the actual cumulative productions from the ideal one is called the total PRVP (abbreviated as TPRVP), see Kubiak (1993). This problem has been widely investigated in the literature since it has a model with a strong mathematical base and wide real-world applications, see Dhamala and Khadka (2009), a recent survey and therein.
In Kubiak (1993), Kubiak solved the TPRVP with a general objective in pseudo-polynomial time $O(D^3)$. The problem is transformed into an equivalent assignment problem. Moreover, several heuristics also exist in the literature for near to optimal solutions, see Dhamala and Khadka (2009). In this paper, we propose a lower and an upper bound for TPRVP. We also establish an explicit lower bound of the problem.
The remainder of the paper is as follows. In the second section, we present a non-linear integer programming formulation. In the third section, we estimate the largest and the smallest function values of a feasible solution of the problem which is the major contribution of this paper. First, the level curves are investigated, then the largest function value and finally the smallest function value. The last section concludes the paper.

NON-LINEAR INTEGER PROGRAMMING FORMULATION
Let D be the total demand of n different models with d_i copies of model $i, i = 1, 2, \cdots, n,$ where $n \geq 2$ and $D = \sum_{i=1}^{n} d_i$. The time horizon is partitioned into D equal time units under the assumption that each copy of a model $i, i = 1, \cdots, n,$ has equal processing time. A copy of a model is produced in a time unit $k, k = 1, \cdots, D,$ means that the copy of the model is produced during the time period from $k - 1$ to k. Let $r_i = \frac{d_i}{D}$ be the demand rate. Let x_{ik} and kr_i be the actual and the ideal cumulative productions, respectively, of model i produced during the time units 1 through k. An inventory holds if $x_{ik} - kr_i > 0$, and a shortage incurs if $kr_i - x_{ik} > 0$. We assign the same cost for both inventory and shortage. Miltenburg (1989) and Kubiak and Sethi (1991) gave an integer programming formulation for TPRVP as follows
with m being a positive integer:

minimize $[F_m = \sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik} - kr_i|^m]$
subject to

$\sum_{i=1}^{n} x_{ik} = k, \quad k = 1,2,\dots,D$

$x_{i(k-1)} \leq x_{ik}, \quad i = 1,2,\dots,n;\ k = 2,3,\dots,D$

$x_{iD} = d_i, x_{io} = 0, \quad i = 1,2,\dots,n$

$x_{ik} \geq 0$, integer $i = 1,2,\dots,n;\ k = 1,2,\dots,D$.

ESTIMATIONS

Level curve

There exist nD deviations between the actual and the ideal cumulative productions of D copies of n models. The value of the actual cumulative production x_{ik}, $i = 1,2,\dots,n;\ k = 1,2,\dots,D$, is sequence-dependent integer from $\{0,1,\dots,d_i\}$. However, the value of the ideal cumulative production kr_i, $i = 1,2,\dots,n;\ k = 1,2,\dots,D$, is sequence-independent rational number. Let j be the number of copies of a model and (i,j) be the j^{th} copy of model $i, i = 1,2,\dots,n$. The actual cumulative production x_{ik}, $i = 1,2,\dots,n;\ k = 1,2,\dots,D$, has nD values with $x_{ik} \in \{j | j = 0,1,2,\dots,d_i;\ i = 1,2,\dots,n\}$. There exist at most $n + D$ different values of x_{ik} for TPRVP. Hence, one can replace x_{ik} by j with $j = 0,1,\dots,d_i;\ i = 1,2,\dots,n$, in the level curve of the objective value of the function of TPRVP. The level curve for copy (i,j) of the objective function of TPRVP is defined as

$f_{ij}{}^m = nD|j - kr_i|^m, i = 1,2,\dots,n;\ j = 0,1,\dots,di$.

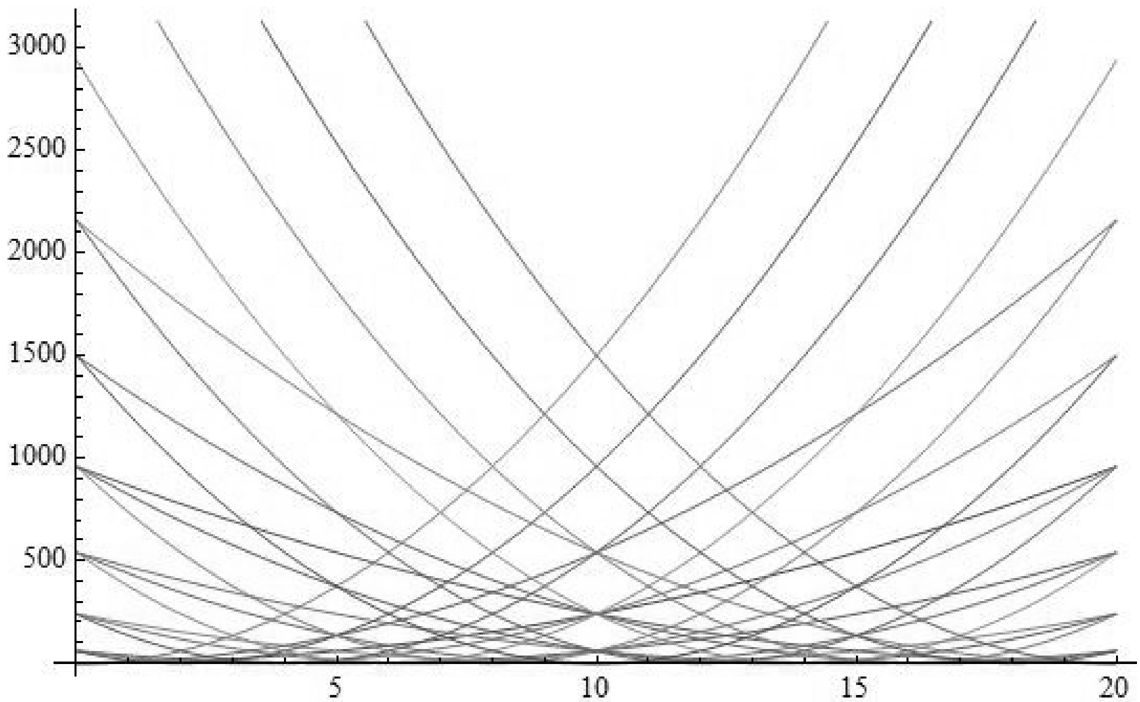

Fig. Level curves $f_{ij}{}^m$ for the instance $(d_1 = 4, d_2 = 6, d_3 = 10)$

Largest function value

We set a horizontal line with a suitable value $B > 0$ intersecting the level curve for each copy $(i,j), i = 1,2,\dots,n;\ j = 1,2,\dots,d_i$, of the objective function of TPRVP on the planning horizon $[0,D]$. The horizontal line with the value B is called a bound for TPRVP. The intersecting points of the level curve of the objective function for each copy and the value B are important to determine the sequencing time units for all copies of all models. A sequencing time unit $k, k = 1,2,\dots,D$, means that a copy of a model is produced during the time units from $k - 1$ to k. One seeks smaller value of B so that the total deviations between actual and the ideal cumulative productions can be reduced with the sequencing time units not exceeding the planning horizon.

It is important to establish the largest and the smallest function values of a feasible solution of the problem so that one can minimize the total deviations in a reasonable time. A sequence corresponding to the minimum value, denoted as B_{min}, which satisfies the inequality

$$\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik} - kr_i|^m \leq B_{min},\ i = 1,2,\dots,n;\ k = 1,2,\dots,D,$$

is optimal for TPRVP.

A necessary and sufficient condition for the existence of a feasible sequence for the product rate variation problem with the objective of minimizing

$max_{i,k}\,|x_{ik} - kr_i|\,, i = 1,2,\dots,n;\ k = 1,2,\dots,D,$ is that the value B_1 must satisfy the two inequalities

$$\sum_{i=1}^{n}(\lfloor k_2 r_i + B_1\rfloor - \lceil (k_1-1)r_i - B_1\rceil) \geq k_2 - k_1 + 1$$

and

$$\sum_{i=1}^{n}(\lceil k_2 r_i - B_1\rceil - \lfloor (k_1-1)r_i + B_1\rfloor) \leq k_2 - k_1 + 1$$

where $k_1, k_2 \in \{1,\dots,D\}, k_1 \leq k_2, [k_1, k_2]$ intersects with the time interval within which copy (i,j) is sequenced, see Brauner and Crama (2004).

Theorem 1: Let B_u be the largest function value of a feasible solution for TPRVP. Then

$$B_u = nD\left(1-\frac{1}{D}\right)^m$$

Proof**:**

Given B_u be the largest function value of a feasible solution for the problem. Then the value B_u satisfies the inequality

$$\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik} - kr_i|^m \leq B_u.$$

Let B_m be the largest function value of a feasible solution for the problem with the objective function $max_{i,k}\,|x_{ik} - kr_i|^m\,, i = 1,2,\dots,n;\ k = 1,2,\dots,D.$ The value B_m satisfies the inequality

$$max_{i,k}\,|x_{ik} - kr_i|^m \leq B_m\,, i = 1,2,\dots,n;\ k = 1,2,\dots,D\,.$$

Consider

$$B_m = \left(1-\frac{1}{D}\right)^m.$$

Then, we can write

$$\left\lfloor k_2 r_i + \sqrt[m]{B_m}\right\rfloor = \left\lfloor k_2 r_i + 1 - \frac{1}{D}\right\rfloor.$$

If $k_2 r_i$ is an integer,

$$\left\lfloor k_2 r_i + 1 - \frac{1}{D}\right\rfloor = k_2 r_i, i = 1,,2,\dots,n\,.$$

and if $k_2 r_i$ is not an integer,

$$k_2 r_i = \lfloor k_2 r_i\rfloor + \in_{i,}\ i = 1,,2,\dots,n\,.$$

where $\in_i$ is the fractional part of $k_2 r_i$.

Since $\frac{1}{D} \leq \in_i \leq 1 - \frac{1}{D}$,

$$\left\lfloor k_2 r_i + 1 - \frac{1}{D}\right\rfloor = \left\lfloor \lfloor k_2 r_i\rfloor + \in_i + 1 - \frac{1}{D}\right\rfloor$$
$$\geq \lfloor k_2 r_i\rfloor + 1$$
$$> k_2 r_i$$

Therefore,

$$\left\lfloor k_2 r_i + \sqrt[m]{B_m}\right\rfloor \geq k_2 r_i$$

Again,

$$\left\lceil k_2 r_i - \sqrt[m]{B_m}\right\rceil = \left\lceil k_2 r_i - 1 + \frac{1}{D}\right\rceil$$

If $k_2 r_i$ is an integer,

$$\left\lceil k_2 r_i - 1 + \frac{1}{D}\right\rceil = k_2 r_i, i = 1,2,\dots,n,$$

and if $k_2 r_i$ is not an integer,

$$\left\lceil k_2 r_i - 1 + \frac{1}{D}\right\rceil = \left\lceil \lfloor k_2 r_i\rfloor - 1 + \frac{1}{D}\right\rceil, i = 1,2,\dots,n,$$
$$\leq \lceil \lfloor k_2 r_i\rfloor \rceil$$
$$< k_2 r_i\,, i = 1,2,\dots,n$$

Therefore,

$$\left\lceil k_2 r_i - \sqrt[m]{B_m}\right\rceil \leq k_2 r_i$$

Hence,

$$\sum_{i=1}^{n}\left(\left\lfloor k_2 r_i + \sqrt[m]{B_m}\right\rfloor - \left\lceil (k_1-1)r_i - \sqrt[m]{B_m}\right\rceil\right)$$
$$\geq \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n}((k_1-1)r_i)$$
$$\geq k_2 - k_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and

$$\sum_{i=1}^{n}\left(\left\lceil k_2 r_i - \sqrt[m]{B_m}\right\rceil - \left\lfloor (k_1-1)r_i + \sqrt[m]{B_m}\right\rfloor\right)$$
$$\leq \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n}((k_1-1)r_i)$$
$$\leq k_2 - k_1 + 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The two inequalities (1) and (2) show that

$$B_m = \left(1-\frac{1}{D}\right)^m$$

is one of the largest function values of a feasible solution for TPRVP with the objective function $max_{i,k}\,|x_{ik} - kr_i|^m\,, i = 1,2,\dots,n;\ k = 1,2,\dots,D.$

Now,

$$\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik} - kr_i|^m$$
$$= \sum_{i=1}^{n}|x_{i1} - 1r_i|^m + \cdots + \sum_{i=1}^{n}|x_{iD} - Dr_i|^m$$
$$= |x_{11} - 1r_1|^m + \ldots + |x_{n1} - 1r_n|^m + \ldots$$
$$+ |x_{1D} - Dr_1|^m + \ldots + |x_{nD} - Dr_n|^m$$
$$\leq nD\,max_{i,k}\,|x_{ik} - kr_i|^m$$
$$\leq nD\left(1-\frac{1}{D}\right)^m$$

Hence, the largest function value B_u of a feasible solution for TPRVP is

$$B_u = nD\left(1-\frac{1}{D}\right)^m.$$

Smallest functions value

If an instance has a feasible sequence at the smallest function value of a feasible solution, the sequence is optimal. However, not all instances are even feasible at this value.

Theorem 2: If B_l be the smallest function value of a feasible solution for TPRVP, then

$$B_l = nD(1 - r_{max})^m .$$

Proof:
Given Bl be the smallest function value of a feasible solution for TPRVP. The
objective function does not exceed Bl. i.e. the inequality

$$B_l \le \sum_{k=1}^{D} \sum_{i=1}^{n} |x_{ik} - kr_i|^m$$

holds.
First copy of a model $i, i = 1, \ldots, n$, must be sequenced at the time unit $k = 1$. So, one can replace x_{ik} by 1 at $k = 1$.
Therefore, for any feasible solution,

$$B_l \le \min D \sum_{i=1}^{n} |1 - r_i|^m$$

Which can be written as

$$B_l \le D[(1 - r_{max})^m + (1 - r_{max})^m + \ldots + (1 - r_{max})^m]$$

The inequality consists of n terms of $(1 - r_{max})^m$.
Thus,

$$B_l \le nD(1 - r_{max})^m .$$

Hence, the smallest function value of a feasible solution for TPRVP is

$$B_l = nD(1 - r_{max})^m.$$

CONCLUSIONS

For the total product rate variation problem, several pseudo-polynomial exact solution algorithms and heuristics have been developed. The largest and the smallest function values of a feasible solution for TPRVP are

$$B_u = nD\left(1 - \frac{1}{D}\right)^m$$

and

$$B_l = nD(1 - r_{max})^m.$$

respectively. These bounds can be used to develop an $O(DlogD)$ exact solution procedure recently given by Khadka and Werner (2014) which improves the known exact algorithm by Kubiak from (1993) with a complexity of $O(D^3)$.

REFERENCES

Brauner, N. and Crama, Y. 2004. The maximum deviation just-in-time scheduling problem. *Discrete Applied Mathematics* **134**: 25-50.

Dhamala, T.N. and Khadka, S.R. 2009. A review on sequencing approaches for mixed-model just-in-time production system. *Iranian Journal of Optimization* **1(3)**:266-290.

Khadka, S.R. and Werner, F. 2014. An exact solution procedure for the total product rate variation problem. *Journal of Manufacturing Systems* Submitted.

Kubiak, W. 1993. Minimizing variation of production rates in just-in-time systems: A survey. *European Journal of Operational Research* **66**:259-271.

Kubiak, W. and Sethi, S. 1991. A note on level schedules for mixed-model assembly lines in just-in-time production systems. *Management Science* **37(1)**:121-122.

Kubiak, W. and Sethi, S. 1994. Optimal just-in-time schedules for flexible transfer lines. *The International Journal of Flexible Manufacturing Systems* **6**:137-154.

Miltenburg, J. 1989. Level schedules for mixed-model assembly lines in just-in-time production systems. *Management Science* **35(2)**:192-207.