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Reverse Holder Condition and Space A_p

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Abstract: In this paper, we begin with Reverse Holder condition and class A_p. We show that a weight function w is in Reverse Holder condition $RH_{p'}(dx)$ if and only if the inverse weight function w^{-1} is in class $A_p(w\,dx)$. Also we show that the weight function w is in $A_p(dx)$ if and only if the inverse weight function w^{-1} is in Reverse Holder condition $RH_{p'}(dx)$.

Keywords: weight function, measure, Reverse Holder condition.

1. Introduction

We begin with some definitions and result which will be used in the proof of our result.

Definition: A locally integrable function on R^n that takes values in the interval $(0,\infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^p(R^n,w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: A function $w(x)\geq 0$ is called an A_1 weight if there is a constant $C_1>0$ such that

$$M(w)(x)\leq C_1\, w(x)$$

where M(w) is uncentered Hardy-Littlewood Maximal function given by

$$M(w)(x) = \sup_{x\in B} \frac{1}{|B|}\int_B w(t)dt.$$

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ is finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\ cubes\ in\ R^n} \left(\frac{1}{|Q|}\int_Q |w(x)|dx\right)\left(\frac{1}{|Q|}\int_Q |w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We note that in the above definition of A_p one can also use set of all balls in R^n instead of all cubes in R^n. Readers are suggested to read [4] for motivation, properties of A_p weights and much more about the A_p weights. Also refer [2] and [3] for more properties on A_1 and A_p weight function.

2. Reverse Holder Condition

Let $1 < q < \infty$ and μ a positive measure on $\boldsymbol{R}^n$. We say that a positive function K on $\boldsymbol{R}^n$ satisfies a reverse Holder condition of order q with respect to measure μ if

$$[K]_{RH_q(\mu)} = \sup_{Q\ cubes\ in\ \boldsymbol{R}^n} \frac{\left(\frac{1}{\mu(Q)}\int_Q K^q\, d\mu\right)^{\frac{1}{q}}}{\frac{1}{\mu(Q)}\int_Q K\, d\mu} < \infty$$

where the supremum is taken on all cubes Q in $\boldsymbol{R}^n$.

Symbolically, we write $K \in RH_q(\mu)$.

We now state and prove our main result.

A weight function w is in Reverse Holder condition $RH_{p'}(dx)$ if and only if the inverse weight function w^{-1} is in class $A_p(w\, dx)$. Also a weight function w is in $A_p(dx)$ if and only if the inverse weight function w^{-1} is in Reverse Holder condition $RH_{p'}(dx)$. Moreover, if a positive function k lies in RH_p (dx) for some $1 < p < \infty$, then there exists $\delta > 0$ such that k lies in $RH_{p+\delta}$ (dx).

Here p and p' are conjugate of each other. So we have

$$p' = 1 - \frac{1}{1-p}\ , 1 - p' = \frac{1}{1-p}, \qquad \frac{1}{p'} = \frac{p-1}{p}$$

Let us write $\int_Q u\, dx = U(Q)$ and $\int_Q v\, dx = V(Q)$.

With this notations we have,

$$I(Q) := \frac{\left(\frac{1}{U(Q)}\int_Q (vu^{-1})^{p'} u\, dx\right)^{\frac{1}{p'}}}{\frac{1}{U(Q)}\int_Q (vu^{-1}) u\, dx}$$

$$= \frac{\left(\frac{V(Q)}{U(Q)}\right)^{\frac{1}{p'}} \left(\frac{1}{V(Q)} \int_Q v^{p'} u^{1-p'} dx\right)^{\frac{1}{p'}}}{\frac{1}{U(Q)} \int_Q v \, dx}$$

$$= \frac{\left(\frac{V(Q)}{U(Q)}\right)^{\frac{p-1}{p}} \left(\frac{1}{V(Q)} \int_Q v^{1-\frac{1}{1-p}} u^{\frac{1}{1-p}} dx\right)^{\frac{p-1}{p}}}{\frac{V(Q)}{U(Q)}}$$

$$= \left(\frac{V(Q)}{U(Q)}\right)^{\frac{p-1}{p}-1} \left(\frac{1}{V(Q)} \int_Q (uv^{-1})^{\frac{1}{1-p}} v \, dx\right)^{\frac{p-1}{p}}$$

$$= \left[\left(\frac{1}{V(Q)} \int_Q (uv^{-1}v) \, dx\right)\left(\frac{1}{V(Q)} \int_Q (uv^{-1})^{\frac{1}{1-p}} v \, dx\right)^{p-1}\right]^{\frac{1}{p}} := J(Q)^{\frac{1}{p}}$$

Note that in the above derivation the following identity was used:

$$\left(\frac{V(Q)}{U(Q)}\right)^{\frac{p-1}{p}-1} = \left(\frac{V(Q)}{U(Q)}\right)^{\frac{1}{p}} = \left(\frac{1}{V(Q)} \int_Q (uv^{-1}) \quad v \, dx\right)^{\frac{1}{p}}$$

Thus we have $I(Q) = J(Q)^{\frac{1}{p}}$. Now

$$[vu^{-1}]_{RH_{p'}(u\,dx)} = \sup \frac{\left(\frac{1}{U(Q)} \int_Q (vu^{-1})^{p'} u \, dx\right)^{\frac{1}{p'}}}{\frac{1}{U(Q)} \int_Q (vu^{-1}) u \, dx} = \sup \ I(Q) = \sup J(Q)^{\frac{1}{p}} = [u \, v^{-1}]_{A_p(vdx)}^{\frac{1}{p}}$$

where supremum is taken over all cubes Q in $\boldsymbol{R}^n$. Let us set $u = 1, v = w, and\ v = 1, v = w$, we have

$$[w]_{RH_{p'}(u\,dx)} = [w^{-1}]_{A_p(wdx)}^{\frac{1}{p}}$$

$$[w^{-1}]_{RH_{p'}(u\,dx)} = [w]_{A_p(dx)}^{\frac{1}{p}}$$

This shows that a weight w is in Reverse Holder condition $RH_{p'}(dx)$ if and only if the inverse weight function w^{-1} is in class $A_p(w\,dx)$. Moreover, a weight function w is in $A_p(dx)$ if and only if the inverse weight function w^{-1} is in Reverse Holder condition $RH_{p'}(dx)$.

3. Conclusion

We established relation between reverse Holder condition and A_p class.

References

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