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Equivalence among the Banach-Alaoglu theorem, Ascoli- Arzela Theorem and Tychonoff's Theorem

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Abstract:
This article is meant to serve as a bridge among these three theorems. The similarity among Banach-Alaoglu theorem, Ascoli-Arzela theorem, and the Tychonoff's theorem are proved. Banach- Alaoglu theorem, Ascoli-Arzela theorem are derived as an immediate consequence of the Tychonoff theorem. These all theorems are compact and equicontinuous and compact Hausdorff spaces are equivalent.

Keywords: Ascoli-Arzela theorem, Alaoglu theorem, Tychonoff's theorem, Product topology

1.Introduction

Banach-Alaoglu theorem, Ascoli-Arzela theorem, and the Tychonoff's theorem are important theorems of topology. Compactness, limit point compactness, and sequential compactness are equivalent for metric spaces. If X is compact in metric then it can be written

$\rho(f,g)$ = max $[d(f(x),g(x))]$ (i)

Compactness is of importance in topology largely, due to its relation with continuity.
These assumptions need additional situation or condition, which is called equicontinuity.

Equicontinuous Function

A family of function is said to be equicontinuous if that functions are continuous and have equal variation over a given neighborhood. The uniform bounded theorem states that a point wise bounded family of continuous linear operators between Banach space is equicontinuous.
A family $\mathcal{F} \subseteq \mathcal{C}([a,b])$ is equicontinuous if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$ for any f $\in \mathcal{F}$ {1}.

Some Review of previous study

The proof of compactness of various subsets of a given metric space is frequently encountered in analysis [1]

Kulkarani [2] studied quantitatively to the Ascoli-

Arzela theorem and declared that a closed and bounded subset of C(X) is nearly compact if and only ifit approaches to equi-continuous. He explained that the stability of this theorem stated as,"a small change in compactness is only a small change in the equicontinuity and vice versa.

AKweittey, Arthur, Appiah, and Dissou [3] had studied Ascoli-Arzela theorem in detail and explained it's application to functional analysis, ordinary differential equation and complex analysis. They used this theorem to simplify the compactness for subsets of spaces of continuous function in much the same way to the Heine-Borel theorem does for subsets of R^n.

Ullrich [4] has derived Ascoli-Arzela theorem with use of Tychonoff theorem. Krukowski, and Mateusz [5] gave a new form of Ascoli-Arzela theorem, which introduces the concepts of equicontinuity along omegas-ultra filters. Mynard [6] derived classical Ascoli-Arzela theorem with the help of continuous convergence and convergence- theoretic techniques.

Rossi [7] explored the new proof of that Banach-Alaoglu theorem and the Tychonoff theorem product theorem for compact Hausdorff spaces are equivalent. Lee explained about possible extension of the Banach-Alaoglu theorem [8]. He showed that if a subset U of a locally convex Hausdorff topological vector space E is at neighborhood which is radial at origin then the polar U is compact of "t-neighborhood" cannot be replaced by "m-neighborhood". Fan [9] generalized the Alaoglu-Bourbaki theorem and explored it's application. Giv[10] expressed the importance of Alaoglu-Bourbaki theorem in the functional analysis. He proved this theorem with the help of existence of the stone-cech compactification for completely regular topological spaces.

Wang [11] proved the compactness of Tychonoff theorem and it's applications. He explained different way to characterize compactness and it's uses in different graph coloring, arithmetic progressions in subsets in $\mathbb{Z}$ in their lecture note.
Pawliuk [12] proved the Tychonoff theorem with help of finite intersection property.

2. Method and Discussion

To show the equivalence among Tychonoff theorem and Banach-Alaoglu theorem,
Banach-Alaoglu theorem is derived from Tychonoff theorem as:

Tychonoff theorem: For each $\lambda \in$ I, $let\ X_\lambda$ be a topological spaces and $\{X_\lambda : \lambda \in \mathrm{I}\}$ is the collection of compact spaces. For each X_λ is compact, then X = $\prod_{\lambda\in \mathrm{I}} X_\lambda$ is compact. In the product topology [12],
Banach- Alaoglu theorem: Let any nor med vector space (X,‖. ‖), also let

$$X^* := \{\mu\colon X \longrightarrow \ | \ l\ \}$$

is complex linear and continuous.

The space X^* is again a linear spaces on which one defining a norm,
$\|\mu\| := Sup_{\|x\|=1}|l(x)|$.

The new normed vector space (X,‖. ‖) is called dual space of (X,‖. ‖).
It makes closed unit ball in X^*. The$n$ B^* is B^* $c$ompact in X^* with respect to $weak^*$ Topology X^*.

$\bar{B}*$ = $\{\mu \in X^* | \|\mu\|| \geq 1\}$[11]
(A)$\Longrightarrow$ (B) Let for each x$\in X,$

$D_X = \{z \in \mathbb{C} : |z| \leq \|x\|\}$ is closed unit ball with radius $\|x\|$ in the complex plane. Now , each D_x is compact in $\mathbb{C}$ then Tychonoff theorem gave D = $\prod_{x\in X} D_X$ is compact in the product topology.

Since $\mu_x \in D_x$, so $\mu = \{\mu_x\}_{x\in X}$ for each x.
Now, μ maps $X\ into\ U_{x\in X}$ D_x = $\mathbb{C}$ which gives $|\mu_x| \leq \|x\|$ for all x$\in X$. So, μ is a functional on X, however it need not be linear. Further writing <x, $\mu >$ = $\mu_x \Rightarrow$ $|< x, \mu >\leq \|x\||$.

If μ is linear$\|\mu\| \ \leq 1$, and $\mu \in B^*$.
Now, for closeness of B^* with respect to product topology.
Let $\{\mu_i\}_{i\in i}$ is a net in B^* and
$\mu_i \rightarrow \ \mu \in D.$
As the canonical projections are continuous in the product topology.
$<$ x ,$\mu_i >= \pi_x(\mu_i) \rightarrow \pi_x$ μ= $<$ x, $\mu >,$ x$\in X$

Now , if x, y $\in X$ and a,b $\in \mathbb{C}$ then,
< a x + b y, μ_i>→< a x + b y, μ>. However, each μ_i is linear and $\mu_i \in B^*,$
< a x + b y, μ_i>= a< x , μ_i> +b < y, μ_i>→ a< x , μ> +b < y, μ> .
< a x + b y, μ> = a< x, μ> +b < y, μ>, so μ is linear and $\mu \in B^*$.
Hence, B^* is a closed subset of D, as D is compact in the product topology [13].

To show the equivalence among Tychonoff theorem and Ascoli-Arzela theorem, Arzela- Ascoli theorem is derived from Tychonoff theorem as:

Ulrich [4] has derived Ascoli –Arzela with Tychonoff theorem in the following way:

(A) (Tychonoff theorem)

Let k is a compact Hausdorff space . Let C(K) is the space of continuous complex-valued functions

on K. Subfamily of $\mathcal{F}$ of C(K) is point wise bounded if for every x in K, there exists r(x)> $0, \forall |f(x)| \leq r(x) for\ all$ f in $\mathcal{F}$. Now $\mathcal{F}$ is equi-continuous if for each in K and $\varepsilon < 0\ there\ exists\ U,\ a\ neighbourhood\ of\ x, \forall |f(x) - f(y)| < \varepsilon$
For all f ∈ $\mathcal{F}$ at any time $y \in U$.

(B) (Ascoli-Arzela theorem)

Let K is a compact Hausdorff space then any point wise bounded and equi-continuous sequence of function in C(K) has a sub sequence converging uniformly on K [3].

Proof:

Let r(x)> 0 for every x∈ K and $\epsilon > 0$. Suppose $\omega(x, \epsilon)$ is the neighborhood of x. Let $\mathcal{F}$ is the family of all functions f : K → ℂ $\forall |f(x)| \leq r(x)$ for all x in K and $|f(x) - f(y)| \leq \varepsilon$ at any time y ∈ $\omega(x, \epsilon)$. Any sequence in a compact metric space has a convergent subsequence , only we have to show that $\mathcal{F}$ is a compact subset of C(K) that is topology of uniform convergence.

Since every element of $\mathcal{F}$ is con$tinue\ \ as\, \mathcal{F} \subseteq C(K)$.
$\mathcal{F} \ni \prod_{x \in K} \overline{D}(0\,, r(x))$ Where $\overline{D}(0\,, r(x))$ is closed disk with centre 0 and radius r(x).
Let $\mathcal{F}_1\,, \mathcal{F}_2 \in \mathcal{F}$ and $\mathcal{F}_1\ and \mathcal{F}_2$ are topology of uniform convergence and topology it inherits as subspace of $\prod_{x \in K} \overline{D}(0\,, r(x))$ respectively [3].
Now we have to show that
(a) $\mathcal{F}_2$ is a closed subspace of $\prod_{x \in K} \overline{D}(0\,, r(x))$
(b) I : $\mathcal{F}_2 \rightarrow \mathcal{F}_1$is continuous.

(a) Assumption of Tychonoff's theorem is $\mathcal{F}_{1,}\ \ \mathcal{F}_2$ are compact.

Let $\mathcal{F}_2$ is a closed subspace of $\prod_{x \in K} \overline{D}\big(0\,, r(x)\big)$ signifies $< f_\alpha >\ \alpha \in A$ is a net in $\mathcal{F}_2$ converges to f in $\prod_{x \in K} \overline{D}\big(0\,, r(x)\big)$ then $f_\alpha(x) \rightarrow f(x)$ for each x in K, so f ∈ $\mathcal{F}_2$.

(b) I : $\mathcal{F}_2 \rightarrow \mathcal{F}_1$is continuous.

Let $\varepsilon > 0$ and K is compact which shows that $x_1, x_2 - - - x_n\ of K,\ \ \forall$ K= $\bigcup_{j=1}^{n} \omega\big(x_j, \epsilon\big)$.
Now, if $< f_\alpha >\ \alpha \in A$ is a net converges to f ∈ $\mathcal{F}_2$ then there exists α_0 in A$\forall \big|f_\alpha(x_j - f(x_j)\big| < \varepsilon$ for all j= 1, 2, 3, ----- n at any time $\alpha \geq \alpha_0$.
For any $\alpha\ and\ x\ in\ K$, choosing j= 1,2,3,------, n $\forall\ x \in \omega\big(x_j, \epsilon\big). we\ get$
$|f_\alpha(x) - f(x\,)\,)| \leq\ \big|f_\alpha(x) - f_\alpha(x_j\,)\big|$+$\big|f_\alpha\big(x_j\ \big) - f x_j\big| + \big|f(x_j\,) - f_x\big| < 3\varepsilon$.
Sup $|f_\alpha - f| < 3\varepsilon$ at any time $\alpha \geq \alpha_{0.}$[11]. This complete the proof.

3. Conclusions

Banach-Alaoglu theorem, Ascoli-Arzela theorem, and the Tychonoff's theorem are equivalent and similar, so they can be derived from each other. The contribution of this study is to meant to understand that all these three theorems are compact and equicontinuous.

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