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Product of A_p Weight Functions

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Abstract: In this paper, we first define A_p weight functions and then show that finite product of weight functions each raised with some power whose sum is one is also an A_p weight function.

Keywords: weight function, Holder's inequality, Maximal function.

1. Introduction

In 1970, Muckenhoupt characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L^p(R^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities. Weighted inequalities are used widely in harmonic analysis. For more about the theory of weights and applications in harmonic analysis, refer [1, 4].

In order to prove the result, some definitions and results are in order:

Definition: A locally integrable function on R^n that takes values in the interval $(0, \infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^p(R^n,w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: A function $w(x) \geq 0$ is called an A_1 weight if there is a constant $C_1 > 0$ such that

$$M(w)(x) \leq C_1 w(x)$$

where M(w) is uncentered Hardy-Littlewood Maximal function given by

$$M(w)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B w(t)dt.$$

If w is an A_1 weight, then the quantity (which is finite) given by

$$[w]_{A_1} = \sup_{Q\ cubes\ in R^n} \left(\frac{1}{|Q|}\int_Q |w(t)|dt\right) ||w^{-1}||_{L^\infty(Q)}$$

is called the A_1 characteristic constant of w.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ is finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\ cubes\ in\ R^n} \left(\frac{1}{|Q|}\int_Q |w(x)|dx\right) \left(\frac{1}{|Q|}\int_Q |w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We remark that in the above definition of A_1 and A_p one can also use set of all balls in R^n instead of all cubes in R^n. Readers are suggested to read [4] for motivation, properties of A_p weights and much more about the A_p weights. Also refer [2] and [3] for more properties on A_1 and A_p weight function.

2. Holder's inequality

Let p and q be two real numbers such that p>1 and $\frac{1}{p}+\frac{1}{q}=1$ and if $f \in L^p$ and $g \in L^q$. Then $f.g \in L^1$ and

$\int fg\, dx \le (\int |f|^p\, dx)^{\frac{1}{p}} (\int |g|^q\, dx)^{\frac{1}{q}}.$

Now we state our main result.

Suppose that weight $w_j \epsilon A_{P_j}$ with $1 \le j \le m$ for some $1 \le p_1 \dots\dots p_m < \infty$ and let $0 < \theta_1 \dots\dots \theta_m < 1$ be such that $\sum_{j=1}^m \theta_j = 1$. We then show that the product function given by

$$W := \prod_{j=1}^m {w_j}^{\theta_j}$$

is an A_p weight function where p is the maximum value of $p_1, \dots\dots, p_m$. The proof will be done in following steps:

(i) We prove that $w_j \epsilon A_P$ for all j.

(ii) We show that the following inequality holds:

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^m {w_j}^{\theta_j}(x)dx \le \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j(x)dx\right)^{\theta_j}$$

(iii) We will use (ii) and the Holder's inequality to show

$$[W]_{A_p} \leq \prod_{j=1}^{m} \left([w_j]_{A_p}\right)^{\theta_j}$$

(iv) Finally by (i) and (iii) we prove that $W \in A_p$.

Since $p_j \leq P$ for all j, using the decreasing nature of w_j, we have $[w_j]_{A_p} \leq [w_j]_{A_{p_j}}$ for all j. This proves (i). To prove (ii) we do as follows. If $w_j = 0$ for some j then the equality holds. Assuming $w_j \neq 0$ for all j and letting $x_j = \frac{w_j(x)}{\frac{1}{|Q|}\int_Q w_j(x)dx}$ one gets

$$\log\left(\prod_{j=1}^{m} x_j^{\theta_j}\right) = \sum_{j=1}^{m} \theta_j \log(x_j) \leq \log\left(\sum_{j=1}^{m} \theta_j x_j\right).$$

We note that we used the concavity of $\log(x)$ in the above expression. Since $\log(x)$ is an increasing function, it follows that

$\prod_{j=1}^{m} x_j^{\theta_j} \leq \sum_{j=1}^{m} \theta_j x_j$.

This implies

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{m} x_j^{\theta_j} dx \leq \frac{1}{|Q|}\int_Q \sum_{j=1}^{m} \theta_j x_j \, dx = \sum_{j=1}^{m} \theta_j \frac{1}{|Q|}\int_Q x_j \, dx = \sum_{j=1}^{m} \theta_j = 1.$$

From the above inequality (ii) follows. Finally we prove (iii).

Let $G := \left(\frac{1}{|Q|}\int_Q Wdx\right)\left(\frac{1}{|Q|}\int_Q W^{\frac{-1}{p-1}}dx\right)^{p-1}$. By (ii) we have,

$$G \leq \left[\prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q |w_j(x)|dx\right)^{\theta_j}\right]\left(\frac{1}{|Q|}\int_Q W^{\frac{-1}{p-1}}dx\right)^{p-1}$$

We write

$W^{\frac{-1}{p-1}} = \prod_{j=1}^{m}\left(\frac{1}{|Q|}\int_Q w_j^{\frac{-1}{p-1}} dx\right)^{\theta_j}$

Let $s = \frac{1}{\theta_1}$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Applying the Holder's inequality, we obtain

$$\int_Q W^{\frac{-1}{p-1}}dx \leq \left(\int_Q w_1^{\frac{-1}{p-1}}(x)dx\right)^{\frac{1}{s}}\left[\int_Q \prod_{j=2}^{m}\left(w_j^{\frac{-1}{p-1}}(x)\right)^{\theta_j s'} dx\right]^{\frac{1}{s'}}$$

$$= \left(\int_Q w_1^{\frac{-1}{p-1}}(x)dx\right)^{\theta_1}\left[\int_Q \prod_{j=2}^{m}\left(w_j^{\frac{-1}{p-1}}(x)\right)^{\frac{\theta_j}{1-\theta_1}} dx\right]^{1-\theta_1}$$

$$\int_Q W^{\frac{-1}{p-1}} dx \leq \prod_{j=1}^{2} \left(\frac{1}{|Q|}\int_Q w_j^{\frac{-1}{p-1}} dx\right)^{\theta_j} \left[\int_Q \prod_{j=3}^{m} \left(w_j^{\frac{-1}{p-1}}(x)\right)^{\frac{\theta_j}{1-\sum_{j=1}^{2}\theta_j}} dx\right]^{1-\sum_{j=1}^{2}\theta_j}$$

Continuing in this manner, one get

$$\int_Q \prod_{j=1}^{m} \left(w_j^{\frac{-1}{p-1}}\right)^{\theta_j} dx \leq C \prod_{j=1}^{m} \left(\int_Q w_j^{\frac{-1}{p-1}} dx\right)^{\theta_j}$$

where C is a constant. Therefore, by (i),

$$G \leq C \prod_{j=1}^{m} \left[\left(\frac{1}{|Q|}\int_Q w_j(x) \, dx\right)\left(\frac{1}{|Q|}\int_Q w_j^{\frac{-1}{p-1}} dx\right)^{P-1}\right]^{\theta_j}$$

Taking supremum over the cube Q in the above inequality, we get $W \in A_{p.}$

References

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