Journal of Advanced College of Engineering and Management, Vol. 10, 2025, Advanced College of Engineering and Management

# A REVIEW ON THE FENCHEL CUTTING PLANE APPROACH TO CAPACITATED FACILITY LOCATION PROBLEM

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## ABSTRACT

The capacitated facility location problem (CFL) is one of the extensively researched challenging problems in combinatorial optimization. CFL consists of deciding which facilities to operate among a set of available locations and how to allocate customers to these opened facilities. Fenchel cuts are a class of cutting planes that solve the separation problem directly with a clear understanding of the polyhedral structure. In this paper, the Fenchel cutting plane method has been used to justify the capacitated facility location problems. A suitable knapsack structure has been chosen to obtain deep cuts using Fenchel cuts. Moreover, a simple heuristic solution is obtained. The comparison of the lower and upper bounds acquired from this method to those subjected to Lagrangian relaxation applied to the demand constraints is reviewed. Specifically, it displayed that the Fenchel cutting planes approach performs better than the Lagrangian one, to obtain bounds and effectiveness when included in a branch or bound algorithm, commencing each relaxation.

**Keywords:** Capacitated facility location problem, Fenchel inequality, cutting plane, Lagrangian relaxation.

## 1. Introduction

The Capacitated facility location (CFL) problem is recognized as one of the extensively researched challenging problems in combinatorial optimization. In CFL, the task involves making decisions about which facilities to operate among a set of available locations and how to allocate customers to these opened facilities. The primary goal of CFL is to minimize the overall fixed costs associated with opening these facilities and satisfying the demands of customers, all while considering the capacity constraints of the facilities. For numerous private and public companies, a crucial decision revolves around determining the optimal locations for their facilities to meet their customer's demands effectively. Beyond its relevance to facility location, CFL finds applications in various contexts, including lot sizing, designing networks, replacing machines, optimizing vehicle routes, and scheduling [1][2].

### 1.1 The CFL problem

Let $F$ be the set of potential facility sites to be opened,$|F| = m$ and $G$ be the set of customers to be allocated to the opened facilities, $|G| = n$. For each of the facilities, $i \in F$ possess a capacity, $u_i > 0$, and each of the customers has a demand $d_j > 0$. To establish a facility at location i, a predefined cost exists, denoted as $f_i \geq 0$, and when assigning customer j to the opened facility i, a cost of $c_{ij} \geq 0$ is incurred.

Let's define the continuous variable $x_{ij}$ as the fraction of the demand of customers' j that is fulfilled by facility i, and $y_i$ be the binary variable defined as:

$y_i = \{1 \quad if\ a\ facility\ is\ opened\ at\ location\ i\ 0 \quad otherwise$

**1.2 Mathematical Model of CFL**

The standard or weak formulation of CFLP is formulated as

$$minimize \sum_{i\in F}\sum_{j\in G} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i \qquad (1)$$

$$subject\ to \sum_{i\in F} x_{ij} = 1\ ,\ \ j = 1, 2, ..., n \qquad (2)$$

$$\sum_{j\in G} d_j x_{ij} \leq u_i y_i\ ,\ \ i = 1, 2, ..., m \qquad (3)$$

$$0 \leq x_{ij} \leq 1\ , \qquad i = 1, 2, ..., m \qquad (4)$$

$$0 \leq y_i \leq 1\ , \qquad j = 1, 2, ..., n \qquad (5)$$

$$y_i \in Z\ , \qquad i = 1, 2, ..., m \qquad (6)$$

The main objective (1) is to select a subset of facilities for opening and determine the allocation of services from these facilities to each client, aiming to minimize the overall costs. Constraints (2) represent the demand restrictions, guaranteeing that the requirements of each customer are met. Constraints (3) represent the capacity constraints that ensure that the demand given to the facility cannot exceed its capacity. Constraints (4 & 5) encompass non-negativity requirements and basic upper bounds on both continuous and discrete variables. If needed, these constraints will be differentiated into $4_x$ and $5_y$. Lastly, constraints (6) refer to the integrality constraints.

## 2. Literature Review

Various methodologies have been applied to address the CFL problem, aiming to establish the most precise achievable lower and upper limits. Lower limits are often found through resolving relaxations, with the Lagrangian relaxation method being the most commonly utilized approach for the CFL issue [2][3][4] and [5][6][7] and [8] for general references.

To establish an upper bound, basic heuristics like modified greedy or interchange heuristic versions have been employed. To address the CFL and other location problems, the Lagrangian heuristics have been used extremely successfully [1][3][9][10] and [11]. In fact, through the resolution of the Lagrangian dual and the determination of the upper limit using the heuristic approach, there is a chance to achieve the same lower bound with the Lagrangian methodology. Using a duality gap, the quality of the limit can be measured [12].

The primary limitation of the prior Lagrangian method becomes evident when integrating the limits into a branch-and-bound procedure for exact problem resolution. Despite a Lagrangian dual's capability to offer a robust lower bound, it is widely recognized that it does not offer a linear programming relaxation of the main problem [12].

Essentially, building on the seminal research of Crowder and co-authors [13], significant lower bounds derived from linear relaxations are fundamentally acquired through cutting plane methods and integrated into a procedure with a branch-and-bound method. These cuts are typically derived from sets of established valid inequalities, necessitating an examination of the facets of the corresponding polyhedral [6] [14]. The advancements in polyhedral techniques, encompassing robust cutting planes and preprocessing methods, have been successfully incorporated into state-of-the-art commercial software packages. Consequently, this integration has significantly expanded the problem sizes that can be effectively addressed.

Several researchers have identified sets of valid inequalities, especially about the CFL issue (refer to [15], [16] and [17], with the latter focusing on cases with constant capacities). These inequalities are typically derived from either pure or mixed structures found within CFL problem formulations, including knapsack, flow, effective capacity, submodular, and single depot inequalities. Aardal [15] has conducted computational experiments involving the implementation of these inequalities in a branch-and-bound algorithm. Notably, after trying out various combinations of inequality sets, Aardal [15] determined that knapsack cover inequalities, derived via the substitute knapsack polytope, were the most impactful. As progress is made through this paper, the utilization of surrogate knapsack structures to obtain both a robust linear relaxation and a heuristic solution through Fenchel cuts will be explored.

The Fenchel cutting planes method facilitates the incorporation of Lagrangian relaxation techniques into polyhedral methods and branch-and-bound algorithms [18]. In this approach, given a formulation and a specific structure denoted as Q, the Fenchel inequalities that describe the convex hull *conv*(Q) are under consideration. These cutting planes are derived straight from the structure Q itself, without requiring any prior understanding of the *conv*(Q) face structure. Boyd [19] developed the Fenchel cutting planes approach, which allows one to solve the convexified issue related to any Lagrangian relaxation, as shown in the work of Saez [18].

In this study, a linear relaxation for CFL issues that matches the value of Lagrangian dual problem was reviewed. Additionally, a primal heuristic method similar to Lagrangian heuristics was studied. This approach allows for comparisons between the Fenchel and Lagrangian methods for CFL problems to be conducted.

## 3. Methodology And Discussion

It is commonly known that solving CFL problem often yields a relatively weak lower limit. Therefore variable upper-bound constraints

$$x_{ij} \leq y_i \quad \forall j \in G, \ \ \forall j = 1, 2, \dots, n \qquad (7)$$

are added to the above problem so that a strong formulation is obtained.

The bound derived from the weak formulation's linear relaxation is acknowledged to be less favorable than the bound resulting from the strong formulation's linear relaxation (weak linear relaxation and strong linear relaxation respectively) [3].

The redundant constraint of total demand

$$\sum_{i \in F} u_i y_i \geq \sum_j d_j \qquad (8)$$

can be added to strengthen the CFL problem's strong formulation.

Various lower bound for CFL is established by partially relaxing or applying a Lagrangian approach to the constraints (2-8). The following notation from Cornuejols et al. [3] is utilized for this purpose. When a set of constraints, such as (2), is completely relaxed, the resulting bound is denoted as $z^2$. Similarly, when constraints other than (2) undergo a Lagrangian relaxation, the resulting associated Lagrangian dual bound is represented as $z_2$.

The collections of equality or inequality constraints are denoted as $(B_1), (B_2), \dots, (B_K)$ . The notation $P(B_1, \dots, B_K)$ is denoted to define the space determined by these constraints$(B_1), (B_2), \dots, (B_K)$.

$$P(B_1, \dots, B_K) = P(B_1) \cap \dots \cap P(B_K).$$

Different treatment is applied to the integrality constraints (8). The feasible convex hull of space characterized by constraints$(B_1), (B_2), \dots, (B_K)$, (8) is represented as $conv(B_1, \dots, B_K, 8)$[3].

Given a mixed integer program D, which is written as

$$\text{(D)} \quad minimize\ cy$$

$$subject\ to\ y \in M$$

where $M = P(B_1, \dots, B_K, 8)$, we have

$$z^{B_i} = min\{cy : y \in M \backslash P(B_i)\}$$

and

$$z_{B_i} = min\{cy : y \in P(B_i) \cap conv(M \backslash P(B_i))\}$$

where the previous equality is obtained from the well-known Geoffrion finding that convexification and dualization are equivalent [20][21]. Convexification transforms the objective function into a convex form, while dualization converts the constraints into a dual problem. Geoffrion's work illustrated that these transformations are interchangeable or equivalent, leading to the same mathematical problem in a different representation. This equivalence is a crucial insight because it allows for alternative problem-solving approaches and provides a theoretical foundation for applying Lagrangian relaxation techniques in optimization problems.

For a given optimization problem (*), the optimal value of problem (*) be denoted as u(*) and its relaxation by$(\overline{*})$. Similarly, for a given structure Q , $\overline{Q}$ represents linear relaxation [18].

Lastly, the formulation is based on the above-notation and is applied as.

$$minimize \sum_{ij} c_{ij} x_{ij} + \sum_{i} f_i y_i$$

$$subject\ to\ (x, y) \in M = P(2, 3, 4, 5, 6, 7, 8)$$

**3.1 Lagrangian Relaxation for the CFL Problem**

Cornuejols et al. [3] explored numerous relaxations in the CFL problem and found that $z_3$ and $z_2$ were more strong, even though it was NP-hard to calculate. The following inequalities are true:

1. $z^{7,6} \leq z^{6} \leq z_3^{8} \leq z_3$

2. $z^{6} \leq z_2 \leq z_3$

3. $z^{7,6} \leq z_3^{7} \leq z_2$

In addition, any inequality is strict for at least one instance of the CFL problem. The bound $z^6$ which is associated with a strong linear relaxation, was significantly improved by $z_3$ and $z_2$.

In the field of research focused on the CFL problem, the relaxation denoted as $z_2$, obtained from dualizing the demand constraints, has gained substantial recognition and usage [1][2][3][4]. The ability to achieve linear programming relaxations with a $z_2$value by combining Lagrangian and Fenchel cutting planes was studied.

The following Lagrangian relaxation must be solved for each multiplier vector $v \in R^n$ to achieve the bound $z_2$.

$$\left(CFL_v\right)\ minimize \sum_{i,j} c_{ij} x_{ij} + \sum_i f_i y_i + \sum_i v_j \left(1 - \sum_i x_{ij}\right)$$

$$subject\ to\ (x, y) \in P(3, 4, 5, 7, 8, 6)$$

Let Q = P (3, 4, 5, 7, 8, 6) be the special structure related to the Lagrangian function L(v) = u($CFL_v$). Both a knapsack problem with continuous variables (n variables) and a knapsack problem with discrete variables (m variables) must be addressed ([1] and [4]) and to calculate L (v).

The Lagrangian dual

$$\left(D_L\right)\ maximize\ L(v)$$

$$subject\ to\ v \in R^n$$

gives the bound $z_2$. In this literature, different methods, such as subgradient and dual ascent, and column generation ([2] [3] [4] and [22]) and have been used to obtain the value $z_2 = u(D_L)$.

**3.2 Adding one Lagrangian inequality**

For any $v \in R^n$ the inequality

$$L(v) \leq \sum_{i,j} c_{ij} x_{ij} + \sum_i f_i y_i + \sum_j v_j \left(1 - \sum_i x_{ij}\right)$$

is considered valid for Q and is referred to as the Lagrangian inequality related to vector v [18].

Shapiro demonstrated that incorporating the above inequality into any linear programming (LP) relaxation guarantees that the objective function's value will be bigger or equal to L(v).

The limit given by the linear relaxation

$$(LP(v))\ \ minimize \sum_{i,j} c_{ij}x_{ij} + \sum_{i} f_i y_i$$

$$subject\ to\ L(v) \le \sum_{i,j} c_{ij}x_{ij} + \sum_{i} f_i y_i + \sum_{j} v_j \left(1 - \sum_{i} x_{ij}\right)$$

$$(x, y) \in P(2, 3, 4, 5, 7, 8)$$

is known to be bigger or equal to L(v) [3]. For any $v \ge 0$ the inequality $z_2 = L(v^*) \le u(LP(v^*))$ holds if $v^*$ is the best solution to $D_L$ [18].

This allows the generation of a linear relaxation with a price that is at least as excellent as $z_2$. Before achieving $LP(v^*)$ by adding a Lagrangian inequality, the Lagrangian dual problem $D_L$ needs resolution. The presence of numerous inequalities here might increase the complexity of solving the relaxation of $LP(v^*)$ and potentially diminish the effectiveness of the subsequent branch-and-bound technique.

### 3.3 Cutting planes from Fenchel for the CFL problem

Fenchel cuts are a type of cutting plane for integer programs that differ from more traditional cuts that attempt to directly solve the separation issue without explicit knowledge of the polyhedral structure of the integer program. i.e. without referring to a specific category of cutting planes, their focus is entirely on the separation problem [19].

For the Problem (T)

$$minimize\ cy$$

$$subject\ to\ By \le b$$

$$y \in Q$$

where $Q \subset R^n$ is nonempty set that contains the integrality constraints.

The primary difficulty in implementing a cutting plane method lies in resolving the separation problem, whether it is related to a group of inequality problems or a convex set encompassing the feasible region [8]. The problem of separating for *conv*(Q), given Q as a particular structure, is solved using the Fenchel cutting plane method.

A piecewise linear concave function called $w(\beta)$ is maximized to produce Fenchel cuts, with the cutting planes being β values for which $w(\beta) > 0$. A cutting plane doesn't exist if the greatest value of $w(\beta)$ is nonpositive, and Fenchel cuts are the deepest cuts that can be produced for a problem in a well-defined sense [19].

**Theorem 1**. Consider $\hat{y}$ as a feasible solution within the linear relaxation of problem P. There exists a value of β for which the value $w(\beta) > 0$ iff there exists a hyperplane $\beta W y \leq f(\beta)$ that separates $\hat{y}$ from the polyhedron $R_F$, where W is the matrix that spans the null space of B. Let us define the $f(\beta)$ and $w(\beta)$ as [19]

$$f(\beta) = \left\{\beta W y : y \in R_F\right\}$$

$$w(\beta) = \beta W \hat{y} - f(\beta)$$

**Observation 1**. When $\beta W \hat{y} \leq f(\beta)$ separates $\hat{y}$ and $R_F$, the distance from $\hat{y}$ to the plane $w(\beta)/||\beta W||$ : when it does not, it is the negative of this distance [12][19].

In conclusion, Fenchel cuts are produced by seeking the maximum value of the function $w(\beta)$ over any domain that contains the origin within its interior and is full-dimensional. If any β results in $w(\beta) > 0$, it signifies the presence of a cutting plane. However, if the greatest value of $w(\beta)$ is zero, it proves the absence of a cutting plane that separates $\hat{y}$ from $R_F$. As a result, after selecting a domain Λ, the problem of separating for *conv*(Q) may be stated as follows [12][19].

$$(R) maximize\ w(\beta)$$

$$subject\ to\ \beta \in \Lambda$$

To take account of a significant number of variables, it is possible to apply several specific domain constraints that will lessen the size of the separation problem.

The equivalence of dualization and convexification, which ensures that $D_L$ is simply the LP dual of the problem

$$\left(D'\right) minimize\ cy$$

$$subject\ to\ By \leq b$$

$$y \in conv(\mathrm{Q})$$

also referred to as the convexified issue for structure Q, is one of the key results obtained from the Lagrangian relaxation theory.

Although Saez [18] demonstrates that the separation theorem may be used after a domain Λ is selected, D' cannot be directly solved in the absence of an explicit description of *conv*(Q).

$$conv(Q) = \{y \in R_F :\ \beta y \leq w(\beta)\ \ \forall\ \beta \in \Lambda\}$$

That is to say, a description of *conv(*Q) is provided by the set of Fenchel inequalities. The consideration of the linear relaxation

$$(LP(Q)) minimize\ cy$$

$$subject\ to\ By \leq b$$

$$y \in \overline{Q}$$

$$\beta y \leq w(\beta) \;\; \forall \;\; \beta \in \Lambda$$

is made possible by this.

The Fenchel cutting planes algorithm will be used to solve what will be referred to as the Fenchel relaxation with respect to structure Q. The choice of structure Q is the most important issue in Fenchel's cutting plane theory. To resolve the problem of separation efficiently, there is a need for structure Q to have these properties [18] [19]:

- Q does not need to satisfy the integrality condition: for instance, it may be the case that *conv*(Q) $\subsetneq \overline{Q}$, because in that case, the Fenchel relaxation's bound and the continuous linear relaxation's bound are identical.
- The optimization shall be relatively simple for Q: remember that in solving the separation problem $w(\beta)$ is to be calculated repeatedly.
- Structure Q must be sparse or separable; otherwise, cuts won't be sparse, and the separation problem will be more challenging due to excessive variables.

**3.4 Fenchel relaxation for the CFL problem**

For the CFL problem, numerous structures to support Fenchel relaxations are chosen. Especially when taking Q = P (2,3,4,5,7,8) a relaxation is obtained,

$$\left(CFL^{*}\right) \, minimize \sum_{i,j} c_{ij} x_{ij} + \sum_{i} f_i y_i$$

$$subject\ to\ (x, y) \in P(2) \cap conv(Q)$$

Keeping in mind that $z_2 = u(CFL^{*})$. It can employ Fenchel cutting planes in place of conventional Lagrangian techniques since $z^6 < z_2$ for at least one case of the CFL problem [3], say *conv*(Q) $\subsetneq \overline{Q}$, does not verify the integrality property.

The equivalence of $CFL^{*}$ to the Fenchel relaxation LP(Q) related to Q is known. A Fenchel cutting planes algorithm could efficiently solve this relaxation provided that the separation problem can be effectively addressed. However, structure Q is not sparse and separable; therefore, a solution to the separation problem is challenging [3].

**Proposition 2**. *conv(Q)* = $P\left(3, 7, 4_x\right) \cap conv\left(8, 5_y, 6\right)$.

This claim allows to replacement of structure Q with the aggregated knapsack structure

$$Q^{K} = \{y \in R^{m}: \sum_{i} u_i y_i \geq \sum_{j} d_j \qquad\qquad y_i \in \{0, 1\} \;\; i = 1, \ldots, m$$

The properties required to solve the CFL* for Fenchel cutting planes have been verified by structure $Q^{K}$.It is crucial to note that the separation issue only contains m variables $\lambda \in R^{n}$ with structure$Q^{K}$, but the separation problem, initialized with starting structure$Q^{K}$, would consist of mn+m variables denoted as$(\mu, \lambda) \in R^{mn+m}$.

Therefore, the bound $z_2$ which is obtained by solving the problem

$$minimize \sum_{i,j} c_{ij}x_{ij} + \sum_i f_i y_i$$

$$subject\ to\ (x, y)\in P(2, 3, 7, 4_x)\cap conv(Q^K)$$

The above problem is the same as Fenchel relaxation, taken with $w(\beta) = maximize(\beta y: y\in Q^K\}$.

$$\left(LP\left(Q^k\right)\right) minimize \sum_{i,j} c_{ij}x_{ij} + \sum_i f_i y_i$$

$$subject\ to\ (x, y)\in P(2, 3, 4, 7, 5)$$

$$\beta y\leq w(\beta)\quad \forall\beta\in\Lambda$$

### 3.5 The separation problem concerning for $conv(Q^K)$

Assume that the CFL problem has a fractional solution, $(\hat{x}, \hat{y})$. A Fenchel cutting plane that can create a separation between $\hat{y}$ and the knapsack polytope $conv(Q^K)$ was searched. Based on established findings for independence systems, an overview of the separation problem for $conv(Q^K)$ was summarized.

**Definition 3**. An $B \subset Z^n_+$ is called an independence system if

i) $0\in S$

ii) $x^a\in B,\ x^b \in Z^n_+\ and\ x^b \leq x^a \Rightarrow x^b\in B$

i.e. $\{x\in Z^n_+ : Ax\leq b$is an independence system if all the coefficients of (A,b) are positive integers[6].

The outcomes are well-known, making the separation problem easier to solve when structure Q is independent, and domain Λ is a unit sphere with an infinity norm $\Lambda_\infty$.

**Theorem 4**. (Saez)[18] Consider an independence system represented by B, a fractional solution denoted as $\hat{y}$ and consider the domain $\Lambda = \Lambda_\infty$ In this case, there exists a β within Λ that maximizes the function $w(\beta)$,and as a result,

*1.* $\beta_i = 0,\ if\ \hat{y}_i = 0.$

*2.* $\beta_i = -1,\ if\ \hat{y}_i = 1.$

*3.* $0\leq\beta_i\leq 1,\ if\ 0 < \hat{y}_i < 1.$

For any other domains, condition 2 does not hold anymore, but conditions 1 and 3 remain.

**Theorem 5**. Let a fractional solution be denoted as $\hat{y}$ of a linear relaxation and a 0-1 aggregated knapsack structure be $Q^K$. Then there exists a β that maximizes the function $w(\beta)$ over the domain $\Lambda_\infty$ such that

*1.* $\beta_i = -1,\ if\ \hat{y}_i = 0.$

*2.* $\beta_i = 0,\ if\ \hat{y}_i = 1.$

*3.* $-1 \le \beta_i \le 0,\ if\ 0 < \hat{y}_i < 1.$

As a consequence, since many variables will be regarded as constants by 1 and 2 conditions, there may be lower variables than m involved in this separation problem. It can accelerate the solution of this separation issue by reducing the domain dimension.

### 3.6 Fenchel heuristic

In contrast to the well-known Lagrangian heuristics [1][2][3][5], a different heuristic approach was introduced. For each iteration of the cutting planes method and each $\beta \in R^m$ the optimal solution of the subproblem $f(\beta) = max\{\beta y : y \in Q^K\}$ at a point denoted as $\bar{y} \in Q^K$ was determined. According to the structure definition $Q^K$, a point $\bar{y}$ satisfies the requirements of the total demand (8), signifying a set of operational facilities with sufficient capacity to meet the total demand. Considering facilities represented by the vector $\bar{y}$, let $\bar{x}$ represent the optimal solution to the subsequent transportation problem:

$$minimize \sum_{i,j} c_{ij} x_{ij}$$

$$subject\ to \sum_i x_{ij} = 1,\ \ j = 1, \dots, n$$

$$\sum_j d_j x_{ij} \le u_i \bar{y}_i, \quad \forall\, i \in I^*$$

where $I^* = \{i \in \{1, \dots, m\} : \bar{y}_i = 1\}$. A feasible solution for the CFL problem is $(\bar{x}, \bar{y})$. The optimal solution out of all those discovered along the cutting planes procedure is used as the heuristic solution.

This suggested heuristic is rather simple to apply, and positive outcomes have been obtained

### 3.7 Benefits of Fenchel relaxation in comparison to Lagrangian relaxation

1. A relaxation based on linear programming is the Fenchel relaxation $LP(Q^K)$. Consequently, once $LP(Q^K)$ was solved to obtain the optimal value, the transition to the branch-and-bound method was possible to determine the overall optimal value. There are two possible solutions to this dual problem with the Lagrangean method, both in solving it and adding an inequality of linear relaxation that is associated with Lagrangian coefficients based on a branch-and-bound algorithm.

1. The dual problem involves n variables, which correspond to the number of clients, as determined through Lagrangian methods. However, in the context of Fenchel cuts, the total number of variables involved in the separation issue is typically limited to the overall number of facilities, denoted as m, as among these some may be sets of constants. Generally, the value of m is considerably smaller compared to n.

## 4. Conclusion

In this review, an attempt has been made to study and provide an overview of the application of the Fenchel cutting plane method for addressing Capacitated Facility Location (CFL) problems. Additionally, the acquisition of a strong linear relaxation and a basic heuristic, known as the Fenchel relaxation and Fenchel Heuristic, has been observed. The superiority of the lower bound obtained from Fenchel relaxation over Lagrangian relaxation has been demonstrated. Similarly, upper bounds have been obtained using both methods.

Furthermore, the benefits of Fenchel relaxation over Lagrangian relaxation become more pronounced when integrated into a branch-and-bound framework based on linear programming for identifying the integer optimal value. Additionally, when employed as an approximation technique, the Fenchel approach yields highly effective solutions.

Exploration of these approximation measures demonstrates that the Fenchel relaxation approach provides solutions that are more effective than those obtained through Lagrangian relaxation.

## 5. Acknowledgements

Sushmita Shrestha acknowledges University Grants Commission (UGC), Nepal for the M.Phil. fellowship (M.Phil-78/79-S & T-05) and Nepal Mathematical Society (NMS) for providing with an NMS-Nick Simon M.Phil. fellowship 2021 for the M.Phil. study.

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