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Some q-Analogues of Hermite-Hadamard Type Integral Inequalities for the Godunova-Levin and s-Godunova-Levin Class of Functions

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Abstract: Hermite and Hadamard independently introduced the Herimite- Hadamard inequality for convex functions for the first time. In the recent years ,varieties of extensions were given with the use of convex functions by the several researchers. In this paper we give q- analogue for the variants of Hermite-Hadamard integral inequality for the Godunova-Levin and s-Godunova class of function.

Key Words: Hermite-Hadamard Inequality, Godunova -Levin function, s-Godunova -Levin function, q-derivative, Jackson q- integration.

1 Introduction and preliminaries

In mathematics,quantum calculus is the study of the classical calculus without the notion of limit and it is also known as q-calculus , where q is a parameter $0 < q < 1$. In q - calculus we obtain mathematical expression in terms of q and whenever $q \to 1$ it again reduces to the original form. The history of q-calculus traced back to the Euler (1707- 1783), who first introduced the q-calculus to deal Newton's work of infinite series. In the twentieth century Jackson [3] was the first mathematician who started the systematic study of q- calculus and introduced q-definite integral .Hermite- Hadamard investigated one of the fundamental inequalities for a convex function in analysis, that is

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

which is known as Hermite-Hadamard inequality. For the first time, in [7], Tariboon and Ntouyas investigated the q-analogue several of classical integral inequalities, from which they

obtained the q-analogue of Hermite - Hadamard inequality. But their finding was not compatible for $q \in (0,1)$ for the left hand side, was proved in [1] by Alp et.al by giving a counter example and proved the correct q- Hermite Hadamard inequality. Recently, many extensions were given with the use of convex functions by several researcher. In 2020 years, the investigation on q- Hermite -Hadamard inequality for general convex functions has been done and several extensions and variants have been developed .
The purpose of this paper is to present the q- calculus analogue of Hermite-Hadamard inequalities for sevral Godunova -Levin class of function in finite interval [a, b].

We now present some notations and definitions from the q-calculus, which are necessary for understanding this paper. Let $J := [a,b] \subset \mathbb{R}$ be an interval and q be a constant with $0 < q < 1$.

Definition 1. *[6] The q-derivative of a continuous function $f : J \to \mathbb{R}$ at x is defined as:*

$$_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)};\ for\ x \neq a \tag{2}$$

For $x = a$ it is defined as

$$_aD_q f(a) = \lim_{x \to a}\ _aD_q f(x)$$

If $_aD_q f(x)$ exists for all $x \in J$, then f is q- differentiable on J. Moreover, if $a = 0$, then 2 reduces to

$$_0D_q f(x) = D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x};\ x \neq 0$$

For more details, see [4]
The higher -order q-derivatives of functions on J are also defined.

Definition 2. *[6] For a continuous function $f : J \to \mathbb{R}$, the second - order derivative of f on J, if $_aD_q f$ is q- differentiable on J, denoted by $_aD_q^2 f$ and defined by*

$$_aD_q^2 f =\ _aD_q(_aD_q)f$$

Similarly, n^{th} order q- derivative $_aD_q^n f$ can be defined on J, provided that $_aD_q^{n-1} f$ is defined on J.

Definition 3. *[6] Let $f : J \to \mathbb{R}$ be a continuous function. Then the q-definite integral on J is represented as*

$$\int_a^x f(t)\ _ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q)a)\ ; for\ x \in J. \tag{3}$$

If a=0 in 3 , it reduces to the classical q-integral called Jackson's q-integral on [0, x] delineated as

$$\int_0^x f(t)\ {}_0d_qt = \int_0^x f(t)\ d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n)\ ; for\ x \in [0,\infty) \tag{4}$$

Theorem 1. *[6] Assume that function $f : J \to \mathbb{R}$ is continuous. Then, we have the following*

(i) ${}_aD_q \int_a^x f(t)\ {}_ad_qt = f(x) - f(a)$;

(ii) $\int_c^x\ {}_aD_qf(t)\ {}_ad_qt = f(x) - f(c)$ *for* $c \in (a, x)$

Theorem 2. *[6] Let functions $f, g : J \to \mathbb{R}$ be continuous and $k \in \mathbb{R}$. Then we have the following*

(i) $\int_a^x [f(t) + g(t)]\ {}_ad_qt = \int_a^x f(t)\ {}_ad_qt + \int_a^x g(t)\ {}_ad_qt$;

(ii) $\int_a^x (kf)(t)\ {}_ad_qt = k\int_a^x f(t)\ {}_ad_qt$;

(iii) $\int_a^x f(t)\ {}_aD_qg(t)\ {}_ad_qt = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\ {}_aD_qf(t)\ {}_ad_qt$ *for* $c \in (a, x)$

The proofs of fundamental theorem on integral calculus , linear property and integration by parts in Theorems 1 and 2, see [6].

Definition 4. *[7] For $\alpha \in \mathbb{R} - \{-1\}$, the definite q- integral is given by*

$$\int_a^x (t-a)^\alpha\ {}_ad_qt = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1} \tag{5}$$

From this one can write

$$\int_0^x t^\alpha\ {}_0d_qt = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)x^{\alpha+1} \tag{6}$$

Definition 5 (Godunova class of function Q(I))**.** *[2] A mapping $f : I \to \mathbb{R}$ is said to belongs to Q(I) class of function if it is non- negative and for all $x, y \in I$ and $\lambda \in (0,1)$ satisfies the inequality*

$$f\Big(\lambda x + (1-\lambda)y\Big)\ \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda} \tag{7}$$

Definition 6 (Godunova class of function P(I))**.** *[2] A function $f : I \to \mathbb{R}$ is said to belong to Godunova -Levin type P(I) class of function if it is non- negative and for all $x, y \in I$ and $\lambda \in [0,1]$ satisfies the inequality*

$$f\Big(\lambda x + (1-\lambda)y\Big)\ \le f(x) + f(y) \tag{8}$$

Definition 7 (s-Godunova-Levin class of function $Q_s(C)$)**.** *[5] A function $f : C \subset X \to [0, \infty)$ is said to be s-Godunova -Levin type wity $x \in [0, 1]$, if*

$$f\Big(tx + (1-t)y\Big) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y) \tag{9}$$

*for all $t \in (0, 1)$ and $x, y \in C$ where **C** is a convex set in linear space X. This class of function is denoted by $Q_s(C)$.*

Theorem 3. *[2] Let $f \in \mathbb{Q}(I), a, b \in I$ with $a < b$ and $f \in L_1[a, b]$.Then one has the inequalities*

$$f\Big(\frac{a+b}{2}\Big) \ \leq \ \frac{4}{b-a} \int_a^b f(x)dx \tag{10}$$

and

$$\frac{1}{b-a} \int_a^b p(x) f(x) dx \ \leq \ \frac{f(a) + f(b)}{2} \tag{11}$$

where $p(x) = \dfrac{(b-x)(x-a)}{(b-a)^2}, x \in [a, b]$. The constant in 10 is the best possible.

Theorem 4. *[2] Let $f \in P(I)$, $a, b \in I$ with $a < b$ and f is integrable in [a, b]*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2}{b-a} \int_a^b f(x)dx \leq 2\Big(f(a) + f(b)\Big) \tag{12}$$

Theorem 5. *[5] Let $f \in Q_s(C)$ with $a < b$ and $f \in L_1[a, b]$, **C= [a,b]** , $s \in [0, 1]$ then one has the inequalities*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2^{s+1}}{(b-a)} \int_a^b f(x)dx \tag{13}$$

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{1-s}; \ s \in [0, 1) \tag{14}$$

2 Main results

Theorem 6 (q-analogue of theorem 3)**.** *Let $f \in Q(I), a, b \in I$ with $a < b$, $0 < q < 1$ and f is integrable in [a,b].Then one has the inequalities*

$$f\Big(\frac{a+b}{2}\Big) \ \leq \ \frac{4}{b-a} \int_a^b f(x)_a d_q x \tag{15}$$

and

$$\frac{1}{b-a}\int_a^b p(x)f(x)\ {}_a d_q x \quad \leq \quad \frac{f(a)+f(b)}{2} \tag{16}$$

where $p(x) = \dfrac{(b-x)(x-a)}{(b-a)^2}, x \in [a,b]$. *The constant in 15 is the best possible.*

Proof. Since $f(x) \in Q(I)$.We have for all $x, y \in I$ with $\lambda = \frac{1}{2}$ and using 7

$$2\Big(f(x)+f(y)\Big) \geq f\Big(\frac{x+y}{2}\Big)$$

Let $x = ta + (1-t)b$ and $y = (1-t)a + tb$ for $t \in [0,1]$.
Then

$$2\Big[f\Big(ta+(1-t)b\Big) + f\Big((1-t)a+tb\Big)\Big] \quad \geq \quad f\Big(\frac{a+b}{2}\Big) \tag{17}$$

Now, q-integrating over t in [0,1].

$$2\int_0^1 f\Big(ta+(1-t)b\Big)\ {}_0d_qt + 2\int_0^1 f\Big((1-t)a+tb\Big)\ {}_0d_qt \geq \int_0^1 f\Big(\frac{a+b}{2}\Big)\ {}_0d_qt \tag{18}$$

Now

$$\begin{aligned}
\int_0^1 f\Big(ta+(1-t)b\Big)\ {}_0d_qt &= (1-q)(1-0)\sum_{n=0}^{\infty} q^n f\Big(q^n a + (1-q^n)b\Big) \\
&= (1-q)\sum_{n=0}^{\infty} q^n f\Big(q^n a + (1-q^n)b\Big) \\
&= (1-q)\frac{(b-a)}{(b-a)}\sum_{n=0}^{\infty} q^n f\Big(q^n a + (1-q^n)b\Big) \\
&= \frac{1}{(b-a)}\int_a^b f(x)\ {}_ad_qx
\end{aligned} \tag{19}$$

But,

$$\int_0^1 f\Big(ta+(1-t)b\Big)\ {}_0d_qt \quad = \quad \int_0^1 f\Big((1-t)a+tb\Big)\ {}_0d_qt$$

Again,

$$\begin{aligned}
\int_0^1 {}_0d_qt &= (1-q)(1-0)\sum_{n=0}^{\infty} q^n \cdot 1 \\
&= (1-q)(1+q+q^2+q^3+\cdots) \\
&= (1-q)\frac{1}{(1-q)} \\
&= 1
\end{aligned} \tag{20}$$

Using above results, one can get

$$\frac{2}{(b-a)}\int_a^b f(x)\ {}_a d_q t + \frac{2}{(b-a)}\int_a^b f(x)\ {}_a d_q t \ \geq\ f(\frac{a+b}{2})$$

$$\therefore f\Big(\frac{a+b}{2}\Big) \ \leq\ \frac{4}{(b-a)}\int_a^b f(x)\ {}_a d_q t \qquad (21)$$

This completes the first inequality and the number 4 is the best.

Again, Since $f \in Q(I)$, for all $a, b \in I$ and $\lambda \in [0,1]$ we have by definition

$$\lambda(1-\lambda)f(\lambda a+(1-\lambda)b) \ \leq\ (1-\lambda)f(a)+\lambda f(b)$$

and

$$\lambda(1-\lambda)f((1-\lambda)a+\lambda b) \ \leq\ \lambda f(a)+(1-\lambda)f(b)$$

Adding and q- integrating over $\lambda \in [0,1]$

$$\int_0^1 \lambda(1-\lambda)f(\lambda a+(1-\lambda)b)\ {}_0d_q\lambda + \int_0^1 \lambda(1-\lambda)f((1-\lambda)a+\lambda b)\ {}_0d_q\lambda \ \leq\ \int_0^1 (f(a)+f(b))\ {}_0d_q\lambda$$

Now,

$$\int_0^1 \lambda(1-\lambda)f((1-\lambda)a+\lambda b)\ {}_0d_q\lambda$$

$$\text{Let } x = (1-\lambda)a+\lambda b$$

q-differentiate with respect to λ, then we get

$$\begin{aligned} {}_0D_q x &= -a+b \\ {}_0D_q x &= b-a \\ \frac{{}_0d_q x}{{}_0d_q\lambda} &= b-a \\ \therefore \frac{1}{b-a}\ {}_0d_q x &= {}_0d_q\lambda \end{aligned}$$

As $\lambda = 0$, then $x = a$ and as $\lambda = 1$, then $x = b$

And

$$\begin{aligned} x &= (1-\lambda)a+\lambda b \\ &= a-\lambda a+\lambda b \\ x-a &= (b-a)\lambda \\ \therefore \lambda &= \Big(\frac{x-a}{b-a}\Big) \end{aligned}$$

Again,

$$\begin{aligned} 1-\lambda &= 1-\frac{x-a}{b-a} \\ 1-\lambda &= \Big(\frac{b-x}{b-a}\Big) \end{aligned}$$

So,

$$\int_0^1 \lambda(1-\lambda)f((1-\lambda)a+\lambda b)\ _0d_q\lambda = \frac{1}{b-a}\int_a^b \frac{(x-a)(b-x)}{(b-a)^2}f(x)\ _ad_qx \tag{22}$$

Similarly one can get

$$\int_0^1 \lambda(1-\lambda)f(\lambda)a+(1-\lambda)b)\ _0d_q\lambda = \frac{1}{b-a}\int_a^b \frac{(x-a)(b-x)}{(b-a)^2}f(x)\ _ad_qx \tag{23}$$

Hence we have all together

$$\begin{aligned} \frac{2}{b-a}\int_a^b \frac{(x-a)(b-x)}{(b-a)^2}f(x)\ _ad_qx &\leq \int_0^1 (f(a)+f(b))\ _0d_q\lambda \\ \therefore \frac{1}{b-a}\int_a^b \frac{(x-a)(b-x)}{(b-a)^2}f(x)\ _ad_qx &\leq \frac{f(a)+f(b)}{2} \end{aligned}$$

This proves the second inequality. □

Remarks 1. *Hermite Hadmard type inequality for the functions in Q(I) are same for q-integral and Riemann integrals. But in this case the result is more sharp in Q(I)space for q-Hadamard inequality.*

Now we give an q- analogue of 12

Theorem 7. *Let $f \in P(I)$, $a, b \in I$ with $a < b$, $0 < q < 1$ and f is integrable in [a, b]*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2}{b-a}\int_a^b f(x)_a d_qx \leq 2\Big(f(a)+f(b)\Big) \tag{24}$$

Proof. As, $f : I \to \mathbb{R}$ belongs to P(I) class, so for all $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f\Big(\lambda x+(1-\lambda)y\Big) \leq f(x)+f(y) \tag{25}$$

Let

$$\begin{aligned} x &= at+(1-t)b \\ y &= (1-t)a+tb \end{aligned}$$

and $\lambda = \frac{1}{2}$ one can get

$$f\Big(\frac{a+b}{2}\Big) \leq f\Big(at + (1-t)b\Big) + f\Big((1-t)a + tb\Big) \tag{26}$$

q- integrating over $t \in [0,1]$, we get

$$\int_0^1 f\Big(\frac{a+b}{2}\Big)_0 d_q t \leq \int_0^1 f\Big(at + (1-t)b\Big)_0 d_q t + \int_0^1 f\Big((1-t)a + tb\Big)_0 d_q t \tag{27}$$

From 19 and 20 the equation 27 becomes

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\leq \frac{1}{b-a}\int_a^b f(x)_a d_q x + \frac{1}{b-a}\int_a^b f(x)_a d_q x \\ f\Big(\frac{a+b}{2}\Big) &\leq \frac{2}{b-a}\int_a^b f(x)_a d_q x \end{aligned} \tag{28}$$

Again, Let $x = a$ and $y = b$ then from 32 one can have

$$f\Big(a\lambda + (1-\lambda)b\Big) \leq f(a) + f(b)$$

Now, q-integrating over $\lambda \in [0,1]$, we get

$$\int_0^1 f\Big(a\lambda + (1-\lambda)b\Big)_0 d_q \lambda \leq \int_0^1 f(a)_0 d_q \lambda + \int_0^1 f(b)_0 d_q \lambda \tag{29}$$

Let us compute the integral in 29

$$\begin{aligned} \int_0^1 f\Big(a\lambda + (1-\lambda)b\Big)_0 d_q \lambda &= (1-q)(1-0)\sum_{n=0}^{\infty} q^n f\Big(aq^n + (1-q^n)b\Big) \\ &= (1-q)\sum_{n=0}^{\infty} q^n f\Big(aq^n + (1-q^n)b\Big) \\ &= (1-q)\frac{(b-a)}{(b-a)}\sum_{n=0}^{\infty} q^n f\Big(aq^n + (1-q^n)b\Big) \\ &= \frac{1}{(b-a)}\int_a^b f(x)_a d_q x \end{aligned} \tag{30}$$

Using 30 and 20 in 29 one can get

$$\frac{1}{(b-a)}\int_a^b f(x)_a d_q x \leq \Big(f(a)+f(b)\Big)$$
$$\Rightarrow \frac{2}{(b-a)}\int_a^b f(x)_a d_q x \leq 2\Big(f(a)+f(b)\Big) \tag{31}$$

So combining 28 and 31 we can get the result.

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2}{b-a}\int_a^b f(x)_a d_q x \leq 2\Big(f(a)+f(b)\Big) \tag{32}$$

□

Remarks 2. *As* $q \to 1$*, then 32 reduces to 12.*

Theorem 8 (q-analogue of theorem 5)**.** *Let* $f \in Q_s(C)$ *with* $a < b$ *and f is integrable in [a, b], C= [a,b],* $0 < q < 1$ *and* $s \in [0,1]$ *then one has the inequalities*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{2^{s+1}}{(b-a)}\int_a^b f(x)_a d_q x \tag{33}$$

$$\frac{1}{b-a}\int_a^b f(x)_a d_q dx \leq \frac{1-q}{1-q^{-s+1}}\Big(f(a)+f(b)\Big);\ s \in [0,1) \tag{34}$$

Proof. Since $f \in Q_s(C)$, we have for all x,y in C with $t = \frac{1}{2}$

$$f\Big(\frac{x+y}{2}\Big) \leq 2^s f(x) + 2^s f(y)$$
$$f\Big(\frac{x+y}{2}\Big) \leq 2^s\Big(f(x)+f(y)\Big) \tag{35}$$

Let

$$x = at + (1-t)b$$
$$y = (1-t)a + tb$$

Then 35 gives

$$f\Big(\frac{a+b}{2}\Big) \leq 2^s\Big[f\Big(ta+(1-t)b\Big) + f\Big((1-t)a+tb\Big)\Big] \tag{36}$$

q-integrating 36 over $t \in [0,1]$, we get

$$\int_0^1 f\Big(\frac{a+b}{2}\Big)_0 d_q t \leq 2^s\Big[\int_0^1 f\Big(ta+(1-t)b\Big)_0 d_q t + \int_0^1 f\Big((1-t)a+tb\Big)_0 d_q t\Big] \tag{37}$$

Now,

$$\int_0^1 f\Big(at+(1-t)b\Big)\,{}_0d_qt = (1-q)(1-0)\sum_{n=0}^{\infty} q^n f\Big(aq^n+(1-q^n)b\Big)$$
$$= (1-q)\sum_{n=0}^{\infty} q^n f\Big(aq^n+(1-q^n)b\Big)$$
$$= (1-q)\frac{(b-a)}{(b-a)}\sum_{n=0}^{\infty} q^n f\Big(aq^n+(1-q^n)b\Big)$$
$$= \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx \tag{38}$$

Also,

$$\begin{aligned}\int_0^1 {}_0d_qt &= (1-q)(1-0)\sum_{n=0}^{\infty} q^n\cdot 1\\ &= (1-q)(1+q+q^2+q^3+\cdots)\\ &= (1-q)\frac{1}{(1-q)}\\ &= 1\end{aligned} \tag{39}$$

So, using 38 and 39 in 37 we get

$$f\Big(\frac{a+b}{2}\Big) \le 2^s\Big[\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx + \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\Big]$$
$$= 2^s\times 2\ \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx$$
$$= \frac{2^{s+1}}{(b-a)}\int_a^b f(x)\,{}_ad_qx \tag{40}$$

So, this proofs first inequality 33

Now we prove the next inequality

As $f\in Q_s(C)$, we have

$$f\Big(ta+(1-t)b\Big) \le \frac{f(a)}{t^s} + \frac{1}{(1-t)^s}f(b)$$
$$f\Big(ta+(1-t)b\Big) \le t^{-s}f(a) + (1-t)^{-s}f(b)$$

q-integrating over $t\in[0,1]$ we get

$$\int_0^1 f\Big(ta+(1-t)b\Big)_0 d_qt \leq \int_0^1 t^{-s} f(a)_0 d_qt + \int_0^1 (1-t)^{-s} f(b)_0 d_qt \tag{41}$$

Now,

$$\begin{aligned}\int_0^1 t^{-s}{}_0 d_qt &= (1-q)\sum_{n=0}^{\infty} q^n (q^n)^{-s} \\ &= (1-q)\sum_{n=0}^{\infty} q^n (q)^{-ns} \\ &= (1-q)\sum_{n=0}^{\infty} q^{n(-s+1)} \\ &= (1-q)\times \frac{1}{1-q^{(-s+1)}} \\ &= \frac{1-q}{1-q^{(-s+1)}};\quad s\in(0,1)\end{aligned} \tag{42}$$

Expression 33 is valid for our calculation in usual integration. For
Take $s = \frac{1}{2}$

$$\begin{aligned}\int_0^1 t^{-\frac{1}{2}} dt &= \Big[\frac{t^{-1/2+1}}{-1/2+1}\Big]_0^1 \\ &= 2\Big(\sqrt{t}\Big)_0^1 \\ &= 2\end{aligned} \tag{43}$$

Again

$$\int_0^1 t^{-s}{}_0 dt = \frac{1-q}{1-q^{-s+1}} \tag{44}$$

As $q \to 1$ the right hand side of equation 44 is also 2. For, Using L'Hospital rule.

$$\begin{aligned}\lim_{q\to 1}\frac{1-q}{1-q^{-s+1}} &= \lim_{q\to 1}\frac{0-1}{1-(-s+1)q^{-s+1-1}} \\ &= \lim_{q\to 1}\frac{-1}{1-(-s+1)q^{-s}} \\ &= \lim_{q\to 1}\frac{-1}{(s-1)q^{-s}} \\ &= \frac{1}{1-s}\end{aligned} \tag{45}$$

As $s = \frac{1}{2}$ then

$$\lim_{q\to 1} \frac{1-q}{1-q^{-s+1}} = \frac{1}{1-1/2}$$
$$= 2 \tag{46}$$

Similarly one we can find

$$\int_0^1 (1-t)^{-s}{}_0d_qt = \frac{1-q}{1-q^{-s+1}} \tag{47}$$

From 41 and using above stuffs one can get

$$\int_0^1 f\Big(ta+(1-t)b\Big)_0d_qt \leq \int_0^1 t^{-s}f(a)_0d_qt + \int_0^1 (1-t)^{-s}f(b)_0d_qt$$
$$\frac{1}{(b-a)}\int_a^b f(x)_ad_qx \leq f(a)\Big(\frac{1-q}{1-q^{-s+1}}\Big) + f(b)\Big(\frac{1-q}{1-q^{-s+1}}\Big)$$
$$= \Big(\frac{1-q}{1-q^{-s+1}}\Big)\Big(f(a)+f(b)\Big)$$
$$\therefore \frac{1}{(b-a)}\int_a^b f(x)_ad_qx \leq \Big(\frac{1-q}{1-q^{-s+1}}\Big)\Big(f(a)+f(b)\Big) \tag{48}$$

This completes the proof. □

Remarks 3. *As $q \to 1$, then 48 reduces to 14*

Remarks 4. *Combining 33 and 34 one can get the following inequality.*

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