

# A COMPARATIVE STUDY OF STEADY-STATE BIOHEAT TRANSFER EQUATIONS IN CYLINDRICAL LIVING TISSUE: ANALYTICAL AND FDM SOLUTIONS

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## Abstract

The study of heat transmission in living human tissue is extremely complicated. The tissue temperature distribution in one-dimensional cylindrical living tissue is investigated based on the Pennes bioheat transfer model. Pennes' biothermal heat transfer model is based on the assumptions of constant thermal conductivity, uniform heat generation rate, and insulated cylindrical surfaces. The analytical solution of the model equation will be a complex process and can be difficult to achieve. The numerical method implemented for model equations is the Finite Difference Method (FDM), which provides a faster and more efficient way to solve model equations. The effects of perfusion rate ( $w_b$ ) and metabolic thermogenesis ( $q_m$ ) are discussed using appropriate parameter values. The graph results show that the numerical approximation technique used is reliable and valid to analyze the effect of  $w_b$  and  $q_m$ . Moreover, it is a powerful tool that can be applied to solve problems of different sizes. Furthermore, the comparison between numerical and analytical results confirms the accuracy of the technique.

**Keywords:** Pennes' Bio-heat transfer equation, Blood perfusion, Metabolic heat generation, Finite difference method, Numerical result.

## 1. Introduction

In a living tissue, heat exchange is an extremely complicated process. Multiple mechanisms are involved in the transport of heat in living tissues, including conduction, convection, radiation, metabolism, evaporation, and phase change [1]. Normally, the body's core temperature is  $37^\circ\text{C}$  [2]. Body temperature regulation is a dynamic mechanism. When heat generated exceeds, the body core temperature decreases. Similarly, if heat loss exceeds, the core temperature increases. As a result, changes in core temperature are equally harmful, therefore body temperatures are kept constant. [3]. Bioheat transfer is the study of heat transfer in biological systems, particularly in living organisms. It involves understanding how heat is generated, transported, and dissipated within the body. There are several key components of bioheat transfer [1], [2], [4], [5];

**Metabolic Heat Generation:** Metabolism is the process by which cells produce energy from nutrients. This metabolic activity generates heat as a byproduct, which is one of the primary sources of heat in the human body.

**Blood Perfusion:** Blood perfusion refers to the flow of blood through tissues. It plays a crucial role in heat transfer because blood carries heat from its source (metabolic activity) to other parts of the body for distribution and ultimately heat dissipation.

**Conduction:** Conduction is the transfer of heat through a material or between two materials in direct contact. In biological systems, heat conduction occurs through tissues and fluids, such as from the core body temperature to the skin.

**Convection:** Convection involves the transfer of heat through the movement of fluids (liquids or gases). In the context of bioheat transfer, convective heat transfer occurs when blood circulates through the body, carrying heat from one region to another.

**Radiation:** Radiation is the transfer of heat in the form of electromagnetic waves. In the human body, the skin and other exposed surfaces emit and absorb thermal radiation.

**Evaporation:** Evaporation is the process of converting liquid to vapor. Sweating is a crucial mechanism in the human body to dissipate excess heat, as the evaporation of sweat from the skin takes away heat from the body.

**Thermoregulation:** The body's ability to maintain a relatively constant internal temperature is essential for its proper functioning. Thermoregulation involves the regulation of various physiological processes to keep the body temperature within a narrow range.

These components work together to maintain thermal balance in the body and to ensure that heat is properly distributed and dissipated, preventing overheating or excessive cooling, which could be harmful to the organism. The study of bioheat transfer is crucial in fields such as medicine, biology, and thermal engineering, as it helps in understanding various biological processes and designing technologies for medical treatments and thermal comfort [6].

The first mathematical model of heat transfer in living tissue, known as the bio-heat equation or Pennes bioheat equation, was initially developed by Henry H. Pennes' in 1948 [7]. The equation has been widely used in the medical community to study thermal effects on living tissue. It is used in the analysis of thermal therapies and to develop new treatments for diseases. This equation is the most frequently employed mathematical model in biological living tissue because of its simplicity and reliability [8], [9]. After that, a great deal of study was done that modified Pennes' predictions, particularly for the perfusion term. Other researchers focused on the several uses for bio-heat equations [10]. Although there are modified versions of Pennes equations, researchers prefer to use Pennes bio-heat equations for applications because of how easily it can be applied to a variety of numerical techniques, such as the finite difference method and the finite element method, to handle the complicated geometry of the dermal part [11], [12], [13], [14]. The study is undertaken using the non-fourier law in the bioheat transfer equation to simulate the thermal effects of the human body in various environments to determine the temperature distribution of the human body and calculate the heat transfer coefficients under different conditions [15], [16], [17]. Saxena [18], [19] and other researchers [3], [9], [20] are working using bioheat equation in human physiology. Gurung and et al. [21], [22], [23] are regourously continuing their work in this area in Nepal. The researchers studied how the body is affected by different temperatures and thermal environments, as well as to calculate the amount of heat that is transferred through the body when exposed to different conditions. It is useful in areas such as human physiology, as it can help to predict how different temperatures and environmental conditions can affect the body, and help to develop treatments and interventions to protect people from the adverse effects of extreme temperatures.

A study of the thermal effects of parameters on a cylindrical living tissue is presented in this paper. Only the radial direction steady state model was considered here, disregarding the axial and angular directions. Based on the results of Zhang et al. (2004) [24], the Finite Difference Method [25] solution is obtained graphically and validated with appropriate physical and physiological parameters. In section 2 model formulation with biological meaning is discussed. Analytical solution and numerical solution is derived

in section 3. Section 4, illustrated results and discussion. Finally, in section 5 overall conclusion is discribed.

**2. Model formulation**

The mathematical model used for bio-heat transfer is based on Pennes’ equation. The Pennes bio-heat equation defines thermal behavior using the Fourier classic law, which illustrates the infinite velocity of heat signal diffusion. This equation's primary drawback is that it treats all blood arteries equally, regardless of their size and shape [7], [26], [27]. The Pennes’ model is preferable for the study of heat transfer between blood and tissue which also associates the effect of metabolism and blood perfusion [9], [18]. The Pennes’ model equation is

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + w_b c_b (T_a - T) + q_m \tag{1}$$

where,  $\nabla \cdot (k \nabla T)$  is heat conduction term where heat conduction through the tissue from regions of higher temperature to regions of lower temperature,  $\rho, c, k$  are the tissue density, the specific heat capacity and the thermal conductivity of tissue respectively.  $w_b$  is the blood perfusion rate per unit volume indicating how much blood is flowing through the tissue, Blood's specific heat is denoted by  $c_b$ , heat generated by tissue metabolism is denoted by  $q_m$ , arterial blood's temperature is denoted by  $T_a$ , and tissue temperature is denoted by  $T$ . The one dimensional steady state bio heat transfer mathematical model based on Pennes’ equation for cylindrical living tissue can be defined as;

$$\frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) + \frac{M}{k} (T_a - T) + \frac{q_m}{k} = 0 \tag{2}$$

where,  $M = w_b c_b$  and the boundary conditions are as,

- (i) initial condition at  $r = 0, \frac{dR}{dT} = 0,$
- (ii) boundary condition at  $r = R, -k \frac{\partial T}{\partial r} = h_A (T - T_\infty).$

where  $R$  is the radius of concerned living tissue,  $h_A$  heat transfer coefficient for the effects of both convection and radiation on the surface of the tissue, and  $T_\infty$  is ambient temperature of tissue. The biological meaning of parameters used and its unit is shown in table below.

Table 2.1 Biological meaning of parameters

S.N	Parameters	Biological meaning	Units
1.	$c_b$	the specific heat of blood	J/(kg. <sup>0</sup> C)
2.	$w_b$	the blood perfusion rate	Kg/(s.m <sup>3</sup> )
3.	$k$	thermal conductivity	W/(m.0C)
4.	$q_m$	the metabolic heat generation	W/m <sup>3</sup>
5.	$r$	the radial coordinate	m
6.	$t$	time	second

7.	T	temperature	$^{\circ}\text{C}$
8.	$h_A$	the heat transfer coefficient	$\text{W}/(\text{m}^2 \cdot ^{\circ}\text{C})$
9.	c	the specific heat capacity	$\text{J}/(\text{kg} \cdot ^{\circ}\text{C})$
10.	$T_{\infty}$	the ambient temperature	$^{\circ}\text{C}$

### 3. Solution Techniques

#### 3.1 Analytical Solution

The non-dimensionalisation of equation (2) and its boundary condition with given characteristic quantities [24] can be rewritten as

$$\frac{1}{r^*} \frac{d}{dr^*} \left( r^* \frac{dT^*}{dr^*} \right) - w_b^* T^* + (w_b^* + q_m^*) = 0 \quad (3)$$

and its boundary condition is

$$r=0, \frac{dT^*}{dr^*} = 0 \text{ and } r=1, \frac{dT^*}{dr^*} = -h_A T^*$$

The equation (3) is change into zero order modified Bessel differential equation and compared with the modified Bessel's equation. Then the analytical solution [17] for T is

$$T = T_{\infty} + (T_a - T_{\infty}) \left( 1 + \frac{q_m^*}{w_b^*} \right) \left[ 1 - \frac{I_0(\sqrt{w_b^*} r^*)}{I_0(\sqrt{w_b^*}) + \frac{\sqrt{w_b^*}}{h_A} I_1(\sqrt{w_b^*})} \right] \quad (4)$$

where,  $I_0$  and  $I_1$  are first and second kind Bessel's equation.

#### 3.2 Finite difference method

The finite difference method [25] is a numerical technique used to solve differential equations and partial differential equations. It is used for solving various engineering and scientific problems where analytical solutions are difficult or not feasible to obtain. The finite difference technique allows for the numerical solution of a wide range of problems with complex boundary conditions and nonlinear functions. As the number of nodes increases, the numerical solution will become more accurate. However, the computational complexity also increases, meaning that a balance must be reached between accuracy and computational efficiency in order to obtain the most effective solution. [13], [28], [29]. From the equation (2), we have

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \left( \frac{dT}{dr} \right) + \frac{M}{k} (T_a - T) + \frac{q_m}{k} = 0 \quad (5)$$

Using the Finite difference method in equation (5) as follow in [13] for the boundary condition

at ;

$$4T_i - \left(4 + \frac{h^2M}{k}\right)T_0 + F = 0, \tag{6}$$

at  $i = 1, 2, \dots, R - 1$ ;

$$\left(1 - \frac{1}{2i}\right)T_{i-1} + \left(1 + \frac{1}{2i}\right)T_{i+1} - \left(2 + \frac{h^2M}{k}\right)T_i + F = 0, \tag{7}$$

and for  $i = R$ ;

$$2T_{R-1} - DT_R + E + F = 0, \tag{8}$$

where,  $D = \left[ \left(2 + \frac{Mh^2}{k}\right) + \left(1 + \frac{1}{2R}\right) \frac{2hh_A}{k} \right]$ ,  $E = \left(1 + \frac{1}{2R}\right) \frac{2hh_A T_\infty}{k}$ ,  $F = \frac{h^2}{k} (MT_a + q_m)$ .

The following linear equation system, shown in matrix form, can be obtained from equations (6), (7), and (8):

$$AX = B \tag{9}$$

where,

$$A = \begin{bmatrix} -\left(4 + \frac{h^2M}{k}\right) & 0 & 0 & \dots & 0 \\ 0.5 & -\left(2 + \frac{h^2M}{k}\right) & 1.5 & \dots & 0 \\ 0 & 0.75 & -\left(2 + \frac{h^2M}{k}\right) & 1.25 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 2 & -D \end{bmatrix}, X = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ T_R \end{bmatrix} \text{ and } B = \begin{bmatrix} -F \\ -F \\ -F \\ \vdots \\ -E - F \end{bmatrix}$$

The matrix (9) can compute the temperature distribution profile in dermal part of human living cylindrical tissue.

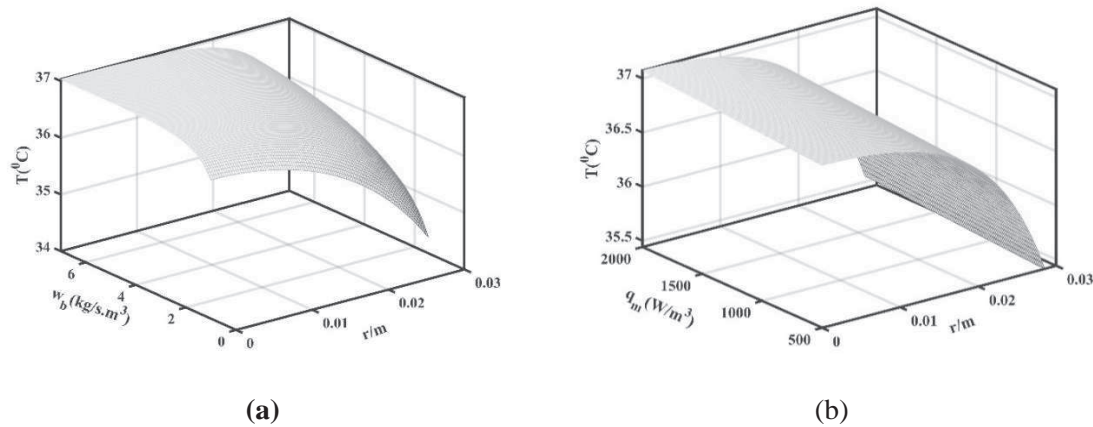
#### 4 . Simulation results and discussion

In this section, the analytic solution obtained from equation (4) and the finite difference solution (9) are used to examine the impact of blood perfusion rate and metabolic heat generation in living tissue. The numerical approximation results effect and the resulting analytic solution are discussed. We use the parameter values taken from [24] with the typical ambient temperature ( $T_\infty$ ) is 25 °C and the number of nodes taken as 100 for the comparative study of the numerical results for the mentioned solution approaches.

Table 4.1: Value of different parameters.

$w_b$ Kg /s.m <sup>3</sup>	$c_b$ J/Kg. °C	$k$ W/ m. °C	$h_A$ W/ m <sup>2</sup> . °C	$q_m$ W/m <sup>3</sup>	$T_a$ °C	$R$ m
3	3850	0.48	10.023	1085	37	0.0285

##### 4.1 Simulation of analytical solution with blood perfusion and metabolic heat generation



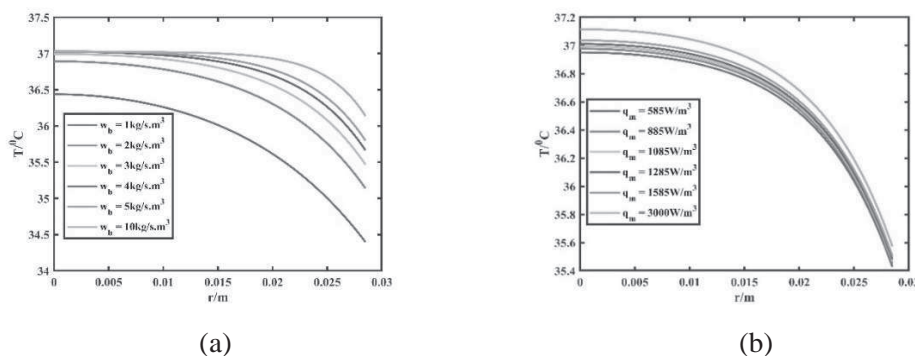
Figures 4.1: Graphical representation of analytical solution and its effects on (a) Blood perfusion rate ( $w_b$ ) and (b) Metabolic heat generation ( $q_m$ ) on tissue temperature.

In Figure 4.1[(a) and (b)], shows the effect of blood perfuction and metabolic heat generation with the radius of cylindrical tissue and variation in temperatue distribution. We have taken the radius of tissue is  $[0, 0.0285]$  metre. In case of  $w_b$  and  $q_m$  the parameter values vary between  $[0, 8]$  kg /s.m<sup>3</sup> and  $[500, 2000]$  w/m<sup>3</sup> with the standard value taken from Table 4.1.

Figure 4.1(a), illustrates how blood perfusion rate responds to changes in tissue temperature. The range starts from a minimum value (low perfusion) and increases toward a maximum value (high perfusion). At low blood perfusion rate, the curve starts at a lower point indicating a relatively low in temperature. As blood perfusion rate increases, the curve gradually ascends, reflecting an increase in tissue temperature rate. When  $w_b = 3$  kg /s.m<sup>3</sup> tissue initial temperature (at  $r = 0$ ) is approximately 37°C. In the same time at outer tissue temperature ( $r = 0.0285$ ) is lower about 34°C. The steepest part of the curve is within the normal tissue temperature range (around 37°C or 98.6°F). As tissue temperature reaches the optimal range, blood vessels dilate, leading to a significant increase in blood perfusion rate. This indicates the highest rate of oxygen and nutrient delivery to the tissue.

Figure 4.1(b), the graph depicts an upward-sloping curve, indicating how tissue temperature responds to changes in metabolic heat generation. At lower tissue temperatures, metabolic heat generation is relatively low. As tissue temperature increases slightly, the metabolic heat generation also starts to rise, but the slope is relatively gentle. Within the normal tissue temperature range (around 37°C or 98.6°F), the curve becomes steeper. As tissue temperature increases, metabolic heat generation increases more rapidly. When  $q_m = 1085$  W/m<sup>3</sup> tissue initial temperature (at  $r = 0$ ) is approximately 37°C. In the same time at outer tissue temperature ( $r = 0.0285$ ) is lower about 34°C. This indicates the body's capacity to generate additional heat in order to keep the body's core temperature stable. As a result, there are fewer blood vessels concentrated on the skin's surface, and the effects of metabolism on the skin's surface is low. The temperature profiles in the figure provide a strong demonstration of this phenomenon. Therefore, it is not possible to detect a substantial variation in the skin's surface temperature as metabolic rate increase; the skin temperature rises nearer the core because metabolism produces heat.

## 4.2 Simulation of finite difference method solution blood perfusion and metabolic heat generation



Figures 4.2: Graphical representation of finite difference method solution and its effects on (a) blood perfusion rate ( $w_b$ ) and (b) metabolic heat generation ( $q_m$ ) on tissue temperature.

Figure 4.2 [(a) and (b)] shows the impact of blood perfusion and metabolic heat generation with tissue temperature in using the numerical finite difference approximation approach. When a different parameter values are used, there is a significant difference in tissue temperature. We found no significant variations in our analysis of tissue temperature distribution in model compared to the analytical graphical representation Figure 4.1 [(a) and (b)]. As a result, this numerical approximation approach would be appropriate for studying the impacts of blood perfusion and metabolic heat generation in the model.

## 5. Conclusion

In the present study, we evaluated various thermal parameters with different blood perfusion and metabolic heat generation parameters of the dermal part of living tissues. Analytic solutions and finite different method solutions are used to compare the impacts of different thermal factors in the cylindrical bio-heat equation.

In the study, the effects of blood perfusion and metabolic heat generation play a significant role in heat transfer models that describe how heat is distributed and exchanged within living tissues. These factors influence temperature distribution, heat dissipation, and overall thermal behavior within biological systems. Blood perfusion refers to the flow of blood through tissues and is a primary mechanism for heat transfer in the body. Blood carries heat from warmer areas to cooler areas, redistributing thermal energy to maintain temperature equilibrium. Metabolic heat generation results from cellular activities and biochemical reactions that produce heat as a byproduct. Because of this process, tissues generate a large amount of heat. Therefore, blood perfusion and metabolic heat generation are integral components of heat transfer models in biological systems. They interact to determine temperature distribution and thermal behavior within tissue and play a vital role in maintaining temperature and supporting various physiological functions.

The numerical solution approaches used to solve the model precisely replicate the analytical results. With the different parametric values, there is a reasonable outcome as a function of thickness, as shown in the figures. Therefore, from three different methods, there is a common conclusion from the simulation results. Moreover, an analytical solution can be challenging or nearly impossible at any time. In conclusion, the study concludes with a comparative analysis of numerical methods for approximation, allowing the reader to select the best numerical method for further research.

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