A BRIEF REVIEW ON MAXIMUM FLOWS IN NETWORKS WITH CONTINUOUS-TIME SETTINGS

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Abstract

Discrete and continuous time dynamic flow problems have been studied for decades. The purpose of the network flow problem is to find the maximum flow that can be sent from the source node to the destination node. Our aim is to review the general class of continuous time dynamic flow problems. We discuss about static cut and generalized dynamic cut, the latter one used to prove the maximum flow minimum cut theorem in continuous case.

Keywords: Cut, generalized cut, flows, reachable node, continuous time dynamic flow

1. Introduction

Nowadays, people focus on instant work with better service and reasonable costs to achieve the desired goal and objectives. As a result, most of the things can be done faster and shorter, as in transportation, production, movement of goods from producer to consumer, communication, information, etc. Network flow models for this purpose present an understanding of the situation at hand therefore; it has been studied over decades. Network flow problems were divided into static and dynamic flow problems. Static network flow problems have been studied for many years and they represent the foundation of network flow problems. Theorems and efficient algorithms related to static network have been developed in the 1950s. Flows over time, also known as dynamic networks, were first introduced by Ford and Fulkerson in [5]. They reviewed the maximum flow problem, which aims to determine the maximum amount of flow that can be sent from a source node to a sink node in a given time horizon. Ford and Fulkerson [4] proved that this problem can be efficiently answered by one minimum cost flow computation on the given network, where transit times of arcs are interpreted as arc costs.

In this paper, we restate the formulation of the maximum dynamic flow problem in continuous time and see the relation between the values of flow and the cut. The rest of the paper is organized as follows. In Section 3, we give basic denotations and necessary concepts. Section 4 is devoted to the static flow and cuts. Section 5 gives a formulation of maximum dynamic flow for continuous time network. In Section 6, we study the maximum flow problem with traversal times, generalized cut and reachable node. With the help of this generalized cut, maximum flow minimum cut theorem in continuous case is explained. Finally Section 7 concludes the paper

2. Literature review

Philpott [8] introduced continuous time flow in a network. He studies the problem of sending as much flow as possible from a source node s to a sink node d in a network with time-varying transit time and storage capacities and also introduced the idea of s-d cuts over time and prove a maximum flow minimum cut theorem. Author in [8] later extended this result to arbitrary transit times. Since then, flow over time have become an active area of research, and many authors are studying various aspects of flows over time in depth. The rate of flow which enters the arc per time unit has been considered to be the flow function. The concept of cuts of source-sink over the time on the network with zero transit time as a solution procedure has been adopted in Anderson et al. [1]. They define a generalized cut and demonstrate that its capacity is an upper bound on the value of any feasible continuous dynamic flow when all transit times are zero. According to [8] this definition and proof can be extended to dynamic flows with non-zero transit times. He proved the maximum flow minimum cut theorem and defined cuts in dynamic networks. The technique used is an analogue of Ford and Fulkerson's labeling algorithm. Fleischer and Tardos [3] investigated them further by transforming a feasible flow in a discrete approach into a feasible flow in a continuous-time setting though natural transformation. Koch et al. [7] and Hashemi and Nasrabadi [6] also explain dynamic flow models in a continuous time setting.

The mathematical formulation for the continuous dynamic contraflow problem was constructed by Pyakurel et al. [9]. They also presented computationally efficient algorithms for solving various dynamic contraflow problems in continuous-time model. Pyakurel et al. [10] investigated efficient continuous contraflow algorithms for evacuation planning problems. This paper focuses on analytical solutions to the continuous-time contraflow problem. They consider the value approximation earliest arrival transshipment contraflow problem for the arbitrary and zero transit times on each arc. Later on Dhamala et al. [2] conducted a detailed survey on network optimization algorithms for evacuation planning problems, where one finds a clear explanation of the network flow approach for evacuation planning problems.

3. Methodology

Basic denotations

Consider a network $N = (V, A, \tau, u, s, d, T)$ with a source $s \in V$ and a sink $\in V$. Here, V represents a finite set of nodes with $|V| = n$, and A represents a finite set of directed arcs $|A| = m$. For all $e \in (i,j)$, the capacity of the arc from node i to node j is represented by u_e. The set of arcs that enter and exit node i are denoted by $\delta^+(i)$ and $\delta^-(i)$, respectively. The cost of the arc e is c_e and the time required to travel the arc $e = (i,j)$ from node i to node j is the transit time τ_e and T represents the total time.

1. Static flow

In two-terminal network N, the maximum static flow (MSF) is defined as follows. Consider the following non-negative function: $f_e: A \rightarrow R^+$ with a static flow of value $|f|$ from the source s to the sink d, satisfying the constraints.

$$\text{Maximize} \quad |f|$$

subject to:

$$\sum_{e \epsilon \delta^+(s)} f_e - \sum_{e \epsilon \delta^-(s)} f_e = -|f| \tag{1}$$

$$\sum_{e\epsilon\delta^+(i)} f_e - \sum_{e\epsilon\delta^-(i)} f_e = 0 \qquad \forall\, i \in V \setminus \{s,d\}, \tag{2}$$

$$\sum_{e\epsilon\delta^+(d)} f_e - \sum_{e\epsilon\delta^-(d)} f_e = |f| \tag{3}$$

$$0 \le f_e \le u_e \qquad e \in A. \tag{4}$$

The flow that leaves the source is represented by equation (1), the flow conservation at the intermediate node is represented by equation (2), the flow towards the sink is represented by equation (3), and the capacity constraint is represented by equation (4). When the cost of flow is taken into account, the cost of each arc is the least.

1. Cut

Definition 1. A cut in network is a partition of nodes such that $X \cup \bar{X} = V, X \cap \bar{X} = \emptyset$, where $s \in X$ and $d \in \bar{X}$. The capacity of cut (s,d) is the sum of all capacities of arcs from s to d denoted by $C(X,\bar{X})$. A cut (s,d) is a minimum cut, if its capacity is the minimum among all s-d cuts.

Theorem 1. [5] "For any network the maximal flow value from source s to sink d is equal to the minimal cut capacity of all cuts separating s and d".

Example 1. Let us consider an example in Figure 1 to verify the maximum flow minimum cut theorem by using Ford and Fulkerson labelling algorithm. Figure 1(a) represents a network having number along with their arc capacity of each arc. Here, we have six paths $P_1 = (sacd)$, $P_2 = (saed)$ $P_3 = (sbed)$, $P_4 = (sbd)$, $P_5 = (sbaed)$, and $P_6 = (sbacd)$ from s to d.

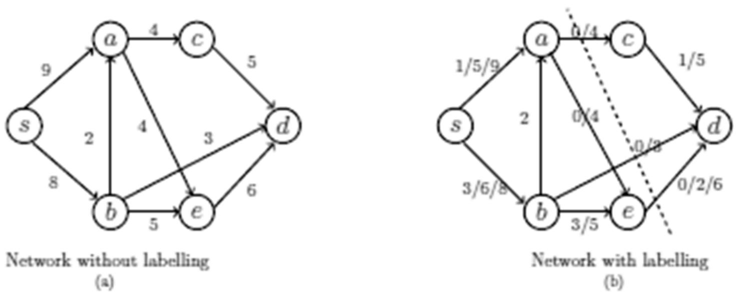

Figure 1: Example of network flow with constant capacity

To calculate the maximum flow minimum cut from s to d, we find the maximum flow minimum cut by using Ford and Fulkerson labelling algorithm as follows.

Step (i) We can send flow from source to sink. Here $4,\ 4,\ 2$ and 3 units flow passes along path P_1, P_2, P_3, and P_4 respectively. Hence, the maximum flow $= 4 + 4 + 2 + 3 = 13$.

Step (ii) We find the minimum cut in given network as

$$\mathrm{X} = \{\mathrm{s}, \mathrm{a}, \mathrm{b}, \mathrm{e}\}, \bar{X} = \{\mathrm{c}, \mathrm{d}\}, \ (\mathrm{X}, \bar{X}) = \{(\mathrm{a}, \mathrm{c}), (\mathrm{b}, \mathrm{d}), (\mathrm{e}, \mathrm{d})\} \text{ with}$$

$$C(\mathrm{X}, \bar{X}) = \mathrm{C}(\mathrm{a}, \mathrm{c}) + \mathrm{C}(\mathrm{b}, \mathrm{d}) + \mathrm{C}(\mathrm{e}, \mathrm{d}) = 4 + 3 + 6 = 13.$$

i.e., minimum cut = 13

Hence, maximum flow = minimum cut = 13.

2. Discrete-time maximum dynamic flow

In this section, we reformulate the maximum dynamic flow problem. Let us consider the discrete time steps as $0, 1, 2, . . ., T-1,\ T.$ The amount of flow is denoted by $f(i, j, t)$ that leaves i along (i, j) at time t, consequently it reaches to the node j at the time $t + \tau(i, j)$. If $\vartheta(T)$ is net flow leaving source s and entering sink d during time steps 0 to 1, 1 to 2, $...T-1$ to T, then the L.P. formulation of maximal dynamic flow problem described by Ford and Fulkerson [5] is

$$\text{Maximize } \vartheta(T)$$

$$\sum_{t=0}^{T} \sum_{(i,j)\in A} [f(i, j, t) - f(j, i, t - \tau(i, j)] = \vartheta(T), \qquad for\ i = s$$

$$\sum_{t=0}^{T} \sum_{(i,j)\in A} [f(i, j, t) - f\big(i, j, t - \tau(i, j)\big)] = 0, \qquad for\ \ i \notin s, d\,; t = 0, 1, \ldots T$$

$$\sum_{t=0}^{T} \sum_{(i,j)\in A} [f(i, j, t) - f\big(j, i, t - \tau(i, j)\big)] = -\vartheta(T), \quad for\ i = d$$

$$0 \le f(i, j, t) \le u(i, j) \quad\ \ for\ all \quad t \in 0, 1, 2 \ldots\ , T \quad and\ \ (i, j) \in A$$

Example 2. Consider the dynamic network depicted in Figure2, which includes a source s, a sink d, and two intermediate nodes a and b. Each arc has capacity and transit time.

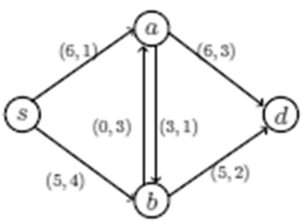

Figure 2: Dynamic Network Flow

3. Continuous-time maximum flow problem

Let $N = (V, A, \tau, u, s, d, T)$ be a network. We take that if (i, j) is an arc of the network then (j, i) is not. For each arc (i, j), we connect a nonnegative real number τ_{ij}, which gives the time to traverse the arc and a nonnegative bounded Lebesgue-measurable function f_{ij} on $[0, T]$, which satisfies capacity constraint. Furthermore, we connect a continuous function a_i on $[0, T]$ to each node i, which gives its

$$\vartheta = \int_0^T \sum_{j=1}^{n} \left[f_{jn}\left(\tau - \tau_{jn}\right) - f_{nj}(\tau)\right] d\tau,$$

storage capacity. For the continuous time maximum flow problem,

where, we connect the convention that $f_{ij}(\tau) = 0$, if either $(i, j) \notin A \; or \; \tau < 0$. We refer to the set of functions $\{f_{ij}(.), (i, j) \in A\}$ as a flow in the network and ϑ as the flow value. The storage $y_i(t)$, in each intermediate node i is defined by

$$y_i(t) = \int_0^t \sum_{j=1}^{n} \left[f_{ji}\left(\tau - \tau_{ji}\right) - f_{ij}(\tau)\right] d\tau, \qquad i = 2, 3, . . ., n - 1. \tag{5}$$

We derive the following formulation of the continuous-time maximum flow problem, also with the activity of the flow in each arc becoming bounded above by its arc capacity and the storage in each node about to be bounded by that node's storage capacity.

$$MFP\text{: } Maximize \qquad \int_0^T \sum_{j=1}^{n} \left[f_{jn}\left(\tau - \tau_{nj}\right) - f_{nj}(\tau)\right] d\tau$$

$$subject\ to \qquad y_i(t) = \int_0^t \sum_{j=1}^{n} \left[f_{ji}\left(\tau - \tau_{ji}\right) - f_{ij}(\tau)\right] d\tau,$$

$$0 \le y_i(t) \le a_i(t), \qquad i = 2, 3, . . . , n - 1, \tag{6}$$

$$0 \le f_{ij}(t) \le u_{ij}(t), \qquad (i, j) \in A, \tag{7}$$

$$t \in [0, T].$$

Generalized cuts

Let N be the network that contains nodes and arcs. A generalized cut in N splitting source s and sink d is defined as [1]:

(i) a set valued function of time ρ is defined on $[0, T]$ with $\rho(t) \subseteq V$, such that for every $t \in [0, T], s \in \rho(t), d \in \bar{\rho}(t)$.

(ii) the set $\gamma_j = \{t: j \in \rho(t)\} \cap (0, T)$ is open for each j in node V.

An open set γ_j can be defined as a countable union of disjoint open intervals, that are unique up to order, from which

$$\gamma_j = \bigcup_{i=1}^{\infty} (\alpha_{ij}, \beta_{ij})$$

i.e., for each node j we can define as

$$A_j = \{ \propto_{ij}: i = 1, 2, 3, \quad . . . \quad \}$$

$$B_j = \left\{ \beta_{ij} : i = 1, 2, 3, . . . \right\}$$

$$C_j = \begin{cases} B_j - \{T\} & if \ T \ \in \ B_j \\ B_j, & otherwise. \end{cases}$$

The value V of the cut ρ in the network N is defined by

$$V = \begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\gamma_j \cap \bar{\gamma}_k} b_{jk}(\tau) d\tau + \sum_{j=2}^{n-1} \sum_{t \in C_j} a_j(t) & if \ \sum_{j=2}^{n-1} \sum_{t \in C_j} a_j(t) \ exists, \\ \infty, & otherwise. \end{cases}$$

Lemma 1. [8] In MFP, the feasible flow value cannot exceed the generalized cut value.

Proof. Let us suppose that f is feasible for MFP with value ϑ. Derived from (5) and the definition of ϑ, we have

$$\vartheta + \sum_{j=2}^{n-1} y_j(T) = \sum_{j=2}^{n} \int_0^T \sum_{k=1}^{n} \left[f_{kj}(\tau - \tau_{kj}) - f_{jk}(\tau) \right] d\tau$$

Define $\Gamma_j = \{t : j \in \ \rho(t)\}$, therefore $\Gamma_1 = (0, T)$

$$\vartheta = \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{0}^{T}[f_{kj}(\tau-\tau_{kj})-f_{jk}(\tau)]d\tau$$

$$= \int_{0}^{T}\sum_{j=1}^{n}\sum_{k=1}^{n}[f_{kj}(\tau)-f_{jk}(\tau)]d\tau - \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{T-\tau_{jk}}^{T} f_{jk}(\tau)d\tau$$

$$+ \int_{\gamma_1}\sum_{k=1}^{n}[f_{1k}(\tau)-f_{k1}(\tau-\tau_{k1})]d\tau - \sum_{j=2}^{n-1} y_j(T). \tag{8}$$

So, we get

$$\vartheta = \int_{\Gamma_1}\sum_{k=1}^{n}[f_{1k}(\tau)-f_{k1}(\tau-\tau_{k1})]d\tau - \sum_{j=2}^{n-1} y_j(T) - \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{T-\tau_{jk}}^{T} f_{jk}(\tau)d\tau\,. \tag{9}$$

According to this equation, the flow into the sink equals the total flow from the source minus that which remains as storage at time T from that which is lost in the case that it gives a node j for a node k later than $T-\tau_{jk}$. Now, even though (5) holds for all $t \in [0,T]$, we can deduce.

$$\int_{\gamma_j}\sum_{k=1}^{n}[f_{kj}(\tau-\tau_{kj})-f_{jk}(\tau)]d\tau = \sum_{i=1}^{\infty}[y_j(\beta_{ij})-y_j(\alpha_{ij})], \tag{10}$$

when the sum on the right exists. Indeed, (10) holds true when the region of integration on its left-hand side is Γ_j, because of integrand (which gives the net instantaneous flow into node j) must be zero whenever $a_j(\tau)=0$, which is true for almost each τ in $\Gamma_j \backslash \gamma_j$. As a result of (10).

$$\sum_{j=2}^{n-1}\int_{\Gamma_j}\sum_{k=1}^{n}[f_{jk}(\tau)-f_{kj}(\tau-\tau_{kj})]d\tau + \sum_{j=2}^{n-1}\sum_{i=1}^{\infty}[y_j(\beta_{ij})-y_j(\alpha_{ij})] = 0 \tag{11}$$

Since $\Gamma_n = \emptyset$ by definition,

$$\int_{\Gamma_n}\sum_{k=1}^{n}[f_{nk}(\tau)-f_{kn}(\tau-\tau_{kn})]\,d\tau = 0. \tag{12}$$

Now, adding (9), (11) and (12) gives,

$$\vartheta = \sum_{j=1}^{n} \int_{\Gamma_j} \sum_{k=1}^{n} \left[f_{jk}(\tau) - f_{kj}(\tau - \tau_{kj})\right] d\tau \\ + \sum_{j=2}^{n-2} \left(\sum_{i=1}^{\infty} \left[y_j(\beta_{ij}) - y_j(\alpha_{ij})\right] - y_j(T) \right) - \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{T-\tau_{jk}}^{T} f_{jk}(\tau) d\tau, \tag{13}$$

when the infinite sums on the right-hand side exist.

Define, $\Gamma_{jk} = \{t : t \in [0, T - \tau_{jk}],\ j \in \rho(t),\ k \notin \rho(t + \tau_{jk})\}$, and $\Lambda_{jk} = \{t : t \in [0, T - \tau_{kj}],\ j \in \rho(t + \tau_{kj}),\ k \notin \rho(t)\}$. The first term on the right-hand side of (13) can be rewritten as

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{jk}} f_{jk}(\tau) d\tau - \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Lambda_{jk}} f_{kj}(\tau) d\tau + \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_j \cap [T-\tau_{jk}, T]} f_{jk}(\tau) d\tau,$$

hence

$$\vartheta = \sum_{j=1}^{n} \sum_{k=1}^{n} \left[\int_{\Gamma_{jk}} f_{jk}(\tau) d\tau - \int_{\Lambda_{jk}} f_{kj}(\tau) d\tau \right] \\ + \sum_{j=2}^{n-1} \left(\sum_{i=1}^{\infty} \left[y_j(\beta_{ij}) - y_j(\alpha_{ij})\right] - y_j(T) \right) - \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{[T-\tau_{jk}, T]/\Gamma_j} f_{jk}(\tau) d\tau, \tag{14}$$

when there are infinite sums on the right-hand side. In this case, since $y_j(t) \geq 0$ and $f_{jk}(t) \geq 0$ for every $t \in [0, T]$, it is easy to derive the following inequality from (14);

$$\vartheta \leq \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{jk}} \boldsymbol{f_{jk}(\tau) d\tau} + \sum_{j=2}^{n-2} \left(\sum_{i=1}^{\infty} \boldsymbol{y_j(\beta_{ij}) - y_j(T)} \right).$$

Consider the case where the right-hand side is infinite. Furthermore, since constraints (6) and (7) ensure that $f_{jk}(t) \leq b_{jk}(t)$ and $y_j(t) \leq a_j(t)$ for every $t \in [0,\ T]$, it follows from the definition of C_j that

$$\vartheta \leq \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{jk}} \boldsymbol{b_{jk}(\tau) d\tau} + \sum_{j=2}^{n-1} \sum_{t \in C_j} \boldsymbol{a_j(t)},$$

which proves the lemma.

Reachable nodes

Consider a feasible flow f_{jk} in network N for MFP. If the storage y_j in node j is defined as equation (5), then the finite application of the recursive rule defines the reachable nodes of N as:

(i) node s is reachable for $t \in [0, T]$.

(ii) if node j is reachable at time t and for $\delta > 0$ such that

$$\underset{t-\delta < t' < t+\delta}{ess\ inf} \{b_{jk}(t') - f_{jk}(t')\} > 0$$

then at time t, k is reachable.

(iii) if node j is reachable at time t and for $\delta > 0$ such that

$$\underset{t-\delta < t' < t+\delta}{ess\ inf} f_{kj}(t') > 0$$

then at time t, k is reachable.

(iv) if node j is reachable at time t_1 and for all $t \in [t_1, t_2)$, $y_j(t) < a_j(t)$, then j is reachable at times $t \in [t_1, t_2)$.

(v) if node j is reachable at time t_1 and for all $t \in (t_2, t_1]$, $y_j(t) > 0$, then j is reachable at all $t \in (t_2, t_1]$.

(vi) if $y_j(T) > 0$, then node j is reachable from source at time T.

Lemma 2. [1] Let $(0, T)$ be a time horizon with a feasible flow in the network N. If the sink is reachable within the time horizon, then the existing flow is strictly less than the feasible flow.

Lemma 3. [1] Let the sink d be not reachable for a feasible flow with in a time horizon $[0, T]$. For a generalized cut by ρ, $\rho(t) = \{j: j\ is\ reachable\ at\ time\ t\ \in [0, T]\}$, the values of flow and cut are equal.

Theorem 2. [1] The values of maximum flow and minimum generalized cuts in a network N are equal, where the cuts separate source and sink.

4. Result and discussion

In this work, we analyzed the continuous dynamic flow problems and its related results. Together with this, it covers the definition of cut and generalized cut with examples. Later, generalized cut and reachable nodes have been used to prove the maximum flow minimum cut theorem in continuous case. The maximum flow minimum cut theorem in continuous case is an analogue of labelling algorithm of Ford and Fulkerson algorithm. Actually, continuous time dynamic flow problems are useful in various fields, like evacuation planning, traffic assignment problem, control theory, etc. These results are of theoretical and practical interests.

5. Suggestion and recommendation

This article researched the maximum flow problem from different aspects, such as tackling various benchmark instances with the use of the most widely adopted approaches followed by its recent real-life applications, and recent utilized approaches for tackling it. Besides, the algorithms based on the network flow were highlighted. For the future work, an interesting point to be considered is to utilize more algorithms. Also, it is crucial to utilize larger networks. Also, since parallel computing has been used recently for enhancing the computation speed of getting a near optimal solution for a variety of real-world optimization problems, parallel maximum flow algorithms and in particular, the coarse-grained approaches can be considered. This is because the coarse-grained approaches have been proven that they outperform other parallel approaches.

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