SOLUTION OF THE BLACK-SCHOLES EQUATION BY FINITE DIFFERENCE SCHEMES

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Abstract

Black-Scholes (BS) equation is a popular mathematical model for determining the value of option in financial derivatives. To predict the option value during the contract of the option is a big problem. Several studies have been shown that the option price value can be determined by applying different methods. In this paper, we have discussed three finite difference methods: Explicit, Implicit and Crank-Nicolson for solving Black-Scholes equation for European call option and compared the obtained results with the exact value. It is found that the Crank-Nicolson method is more accurate and cost effective in comparison with explicit and implicit methods.

Keywords: *Travel Strike Price, Expiration Time, Risk-free Interest Rate, Call, Put*

1. Introduction

The Black-Scholes model is a very popular mathematical model in option pricing theory to determine the price of options. Before 1973, to predict the value of an option was a really big problem. In 1973, Fisher Black and Myron Scholes together addressed this job in a partial differential equation approach publishing a paper *The price of Options and Corporate Liabilities* in the journal: The journal of political economy [1]. At the same time Robert. C Merton also solved the problem in the same direction [8]. For this great contribution Scholes and Merton were awarded by Novel prize in 1997 [12]. During the last years, different studies have provided solutions of the equation applying different analytical and numerical methods [2], [3], [13]. Black and Scholes derived an analytical solution in 1973 [1]. Forsyth et al. (1999) used the finite element approach to the pricing of option [13]. Along similar lines, Tangman et al. (2008) considered High-Order Compact (HOC) schemes to discretize the Black-Scholes PDE for the numerical pricing of European option. Song and Wang (2013) applied symbolic calculation software to provide a numerical solution using the implicit scheme of the finite difference method. Two years later, Uddin et al. (2015) presented the numerical result of semi-discrete and full-discrete schemes for European call option and put option by Finite Difference Method and Finite Element Method [13], [12]. Ankudinova and Ehrhardt (2008) analyzed that the Crank-Nicolson are the most accurate techniques to price the European call option [12]. Darae et al found a numerical solution of Black-Scholes European options without using boundary conditions [5]. A comparative result between fully implicit, Crank-Nicolson finite difference methods and Monte-Carlo method is found in the paper of Nwozo and Fadugba [10] which concluded that the finite difference methods are most accurate and faster than the Monte-Carlo method.

The Black–Scholes partial differential equation introduced by Black and Scholes to determine the value of European call option is in the form

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

With the boundary and final conditions

$$V(0,\ t)=0$$

$$V(S,t)\rightarrow S \text{ as } S\rightarrow+\infty$$

$$V(S,\ T)=\max(S-K,0)$$

where V is the value of an option, r is the risk-free interest rate, □ is the volatility and S is the price of underlying asset at time t. This work focuses on finding the solution of Black-Scholes equation for European call option. A European call option is the right but not obligation to purchase an underlying asset at the exercise price K at the expiration time T. In fact, Options are the tools against the uncertainty of the market. The writer of an option gives its holder right to hedge the risk by limiting the loss, for which the writer is paid a premium called the option price [11]. Our purpose is to find the option price by solving the Black-Scholes equation using three finite difference schemes: explicit, implicit and Crank-Nicolson. We also compare our results with the exact solution.

2. Numerical Solution

A closed form solution can be found transforming the BS equation into a heat equation by applying suitable substitutions. The transformed heat equation is of the form

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2},\ -\infty<x<\infty,\ 0\leq\tau\leq\frac{\sigma^2}{2}T \tag{2}$$

with initial condition

$$u(x,0)=e^{\frac{1}{2}(k+1)x}-e^{\frac{1}{2}(k-1)x}. \tag{3}$$

The closed form solution [4] of the equation (2) for the European call option is given by

$u(S,\ t)=S$ □$(d_1)-K\,e^{-r(T-t)}$□(d_2)

where $\psi(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{\frac{-y^2}{2}}dy$ is the cumulative distribution function for the standard normal distribution and

$$d_1=\frac{\log(S/K)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},d_2=\frac{\log(S/K)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

The numerical values calculated using Matlab program in different cases are given in chapter 3.

3. Discretization of the Equational Mesh

Since we cannot numerically solve the BS equation on an infinite domain $(0,\ \infty)$, we truncate the infinite domain into a finite domain taking stock price sufficiently large [5]. We discretize the Black-Scholes equation (1) by dividing the region $[0,\ T]\times[0,\ S_{\max}]$ into a finite $N\times M$ number of grid points:

$$t_0<t_1<\cdots<t_{\max}=T,\ T=N\text{□}t,$$

$$S_0<S_1<\cdots<S_M,\ S_M=M\text{□}S.$$

We approximate the time derivative $\frac{\partial V}{\partial t}$ by the backward difference approximation

$$\frac{V_{i,j} - V_{i-1,j}}{\Delta t}$$

and the spatial derivatives $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ by the central difference approximations for the case of explicit method. Then the equation (1) reduces in the form

$$\frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S}$$

$$\frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S^2}$$

$$\frac{V_{i,j} - V_{i-1,j}}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \left(\frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S^2}\right) + rS_j \left(\frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S}\right) = rV_{i,j}$$

where $S_j = j\Delta S$. After few steps simplification, we obtain a system of N equations for M variables of the form

$$V_{i-1,j} = a_j V_{i,j-1} + b_j V_{i,j} + c_i V_{i,j+1}, i = N, N-1, \cdots 1; j = 1, 2, \cdots, M-1$$

where

$$a_j = \frac{1}{2}\Delta t(\sigma^2 j^2 - rj)$$
$$b_j = 1 - \Delta t(\sigma^2 j^2 + rj)$$
$$c_j = \frac{1}{2}\Delta t(\sigma^2 j^2 + rj).$$

In matrix form, this can be written as

$$AV_i = V_{i-1}$$

The explicit method is conditionally stable. The time step Δt should necessarily be small because the process is valid only for $0 < \frac{\Delta t}{\Delta S^2} \leq \frac{1}{2}$, that is $\Delta t \leq \frac{1}{2}\Delta S^2$, and ΔS should be kept small in order to attain reasonable accuracy. But this method seems fast since it requires only previous step values to compute current values. The explicit finite method is accurate to $O(\Delta t, \Delta S^2)$. Implicit finite difference methods are used to overcome the stability issue. There is no more need for ridiculously small time-steps. For this method, the time derivative is replaced by the forward difference approximation and the spatial derivatives are approximated by the central difference approximations. The standard system of equations is of the form

$$V_{i,j} = a_j V_{i-1,j-1} + b_j V_{i-1,j} + c_j V_{i-1,j+1}$$

where

$$a_j = \frac{1}{2}\Delta t(rj - \sigma^2 j^2)$$
$$b_j = 1 + \Delta t(\sigma^2 j^2 + r)$$
$$c_j = -\frac{1}{2}\Delta t(rj + \sigma^2 j^2).$$

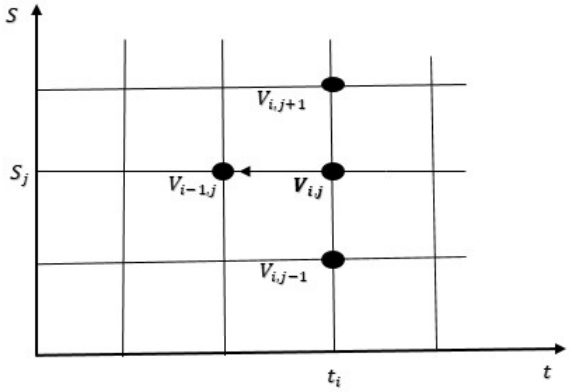

Figure 1: Discretization of Explicit Method

The implicit method is unconditionally stable despite the fact that it requires larger computations since it requires both current and previous step iteration values to compute present values. From one value at time T, three values from time step $T - \Delta t$ should be found, fortunately there are upper and lower boundary conditions, so system of equations is made and it can be solved. This kind of calculating prices spreads backward until step for present time is reached.

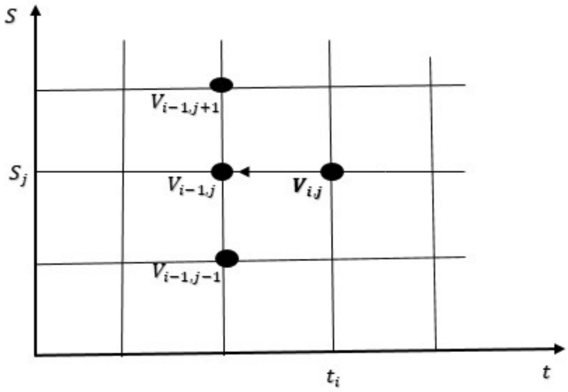

Figure 2: Discretization of Implicit Method

Crank-Nicolson method has been introduced in order to improve accuracy up to $O(\Delta t^2)$, by combining the explicit and implicit methods. This method is unconditionally stable and accurate than the previous methods. For the time derivative the central difference approximation is applied. The standard system of equations is given by

$$-a_j V_{i-1,j-1} + (1 - b_j) V_{i-1,j} - c_j V_{i-1,j+1} = a_j V_{i,j-1} + (1 + b_j) V_{i,j} + c_j V_{i,j+1}$$

where

$$a_j = \frac{\Delta t}{4}(\sigma^2 j^2 - rj)$$
$$b_j = -\frac{\Delta t}{2}(\sigma^2 j^2 + r)$$
$$c_j = \frac{\Delta t}{4}(\sigma^2 j^2 + rj).$$

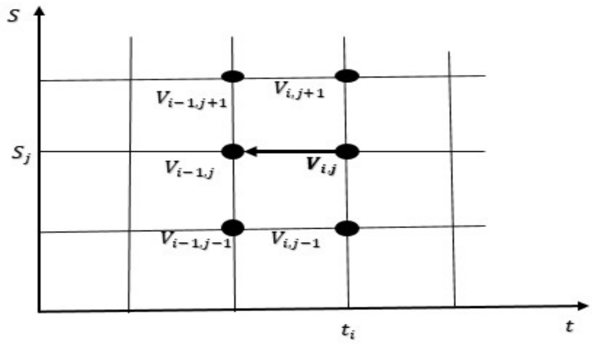

Figure 3: Discretization of Crank-Nicolson Method

$$-a_jV_{i-1,j-1} + (1-b_j)V_{i-1,j} - c_jV_{i-1,j+1} = a_jV_{i,j-1} + (1+b_j)V_{i,j} + c_jV_{i,j+1}$$

where

$$a_j = \frac{\Delta t}{4}(\sigma^2 j^2 - rj)$$
$$b_j = -\frac{\Delta t}{2}(\sigma^2 j^2 + r)$$
$$c_j = \frac{\Delta t}{4}(\sigma^2 j^2 + rj)$$

4. Results and Discussion

We observed three methods taking different time step sizes Δt and stock price step sizes ΔS. The initial price is considered 70 and strike price 85. The risk-free interest is assumed r = 0.1 and the volatility rate is taken σ = 0.2. Figure (4) shows that all three methods seem stable and convergence for respective time step sizes Δt but differed by small error, see table 1. For large time step size Δt the explicit method seems very far from convergence, see figure 5. For Δt = 0.0001 the two methods: explicit and implicit seems very closer in case of convergence but the implicit method takes large amount of computational time, see table 1. The Crank-Nicolson method seems far better than the other two methods because for large step size Δt = 0.01 the method converges to the exact solutions whereas explicit and implicit methods are far from convergence for the small step size Δt = 0.0001 compared to Crank-Nicolson method.

Table 1: The option values calculated by Exact, explicit, implicit and crank-Nicolson methods

		Explicit		Implicit		Crank-Nicolson	
Stock Price	Exact Value	Value	Time(S)	Value	Time(S)	Value	Time(S)
70	3.0296	3.0293	0.930	3.0299	0.777	3.0296	0.128
75	5.1424	5.1419	0.938	5.1418	0.748	5.1421	0.139
80	7.9141	7.9135	0.917	7.9129	0.754	7.9140	0.136
85	11.2792	11.2782	0.931	11.2773	0.769	11.2796	0.136
90	15.1336	15.1326	0.981	15.1316	0.848	15.1341	0.177
95	19.3632	19.3622	1.053	19.3613	0.971	19.3638	0.184
100	23.8635	23.8626	1.097	23.8620	1.111	23.8644	0.206

Taking the same time step □t = 0.0001 it seems that the two methods give the almost same results but the explicit method is faster than the implicit method. The time amount per iteration in implicit method is larger than that of the explicit values.

Table 2: The numerical results by Explicit and Implicit methods taking □t = 0.0001 for both methods.

Stock Price	**Explicit**	**Implicit**	**Time(S)**
70	3.0293	3.0294	7.413
75	5.1419	5.1419	7.315
80	7.9135	7.9135	7.376
85	11.2782	11.2782	7.338
90	15.1326	15.1325	8.218
95	19.3622	19.3622	9.627
100	23.8626	23.8627	10.361

The following graph shows that the three methods converge to the exact values though they represent small error. The explicit values are calculated taking □S = 0.5 and □t = 0.0001 whereas for implicit method □S = 0.5 and □t = 0.001 are taken and for Crank-Nicolson methods we have assumed □S = 0.5 and □t = 0.01. This implies that explicit method requires comparatively small-time step size (large time mesh) for the convergence.

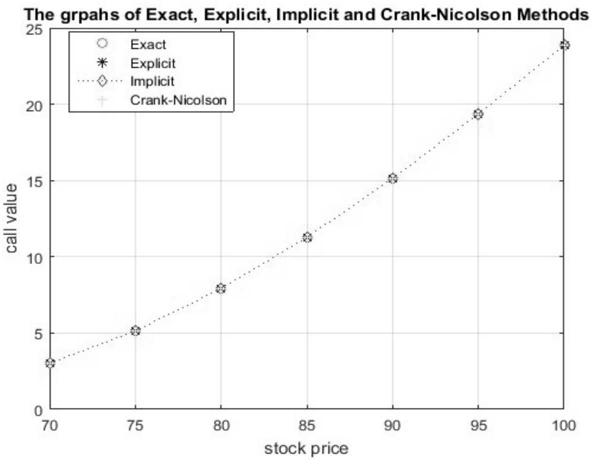

Figure 4: Graphical Representation of Exact, Explicit, Implicit and Crank-Nicolson Values

The following graph shows the divergence character of explicit method for comparatively large step size □t = 0.001.

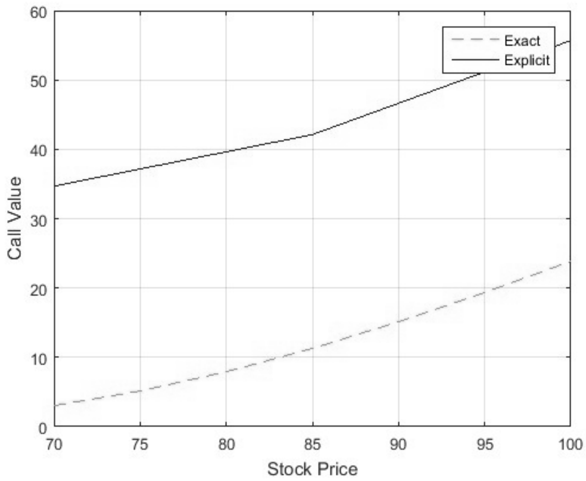

Figure 5: Exact vs Explicit Values

5. Conclusion

We have applied three finite difference methods: explicit, implicit and Crank-Nicolson to solve the Black-Scholes partial differential equation for the European call option and compared the obtained results with the exact solution. To calculate the numerical values, we have used Matlab15 version on our own 64bit core i5 laptop. From the experiment, we observed that the explicit method is fast despite the fact that it requires very large time-price mesh. The implicit method is unconditionally stable but requires large computational time in each calculation. The Crank-Nicolson method is better than these two methods. It is fast and stable too. In future, my interest is to find the numerical solution of the Black-Scholes equation for European option with non-constant volatility and interest rate.

References

1) Black Fisher and Scholes Myron, *"The Pricing of Options and Corporate Liabilities"*, The Journal Political Economy, 1973.

2) Cen Z. and Le A., *"A robust and Accurate Finite Difference Method for a Generalized Black-Scholes Equation,"* Elsevier, 2011.

3) Courtadon Georges, *"A More Accurate Finite Difference Approximation for the Valuation of Options,"* The Journal of Financial and Quantitative Analysis, 1992.

4) D C Ganga Ram, *"Numerical Solution of Black-Scholes Equation by Finite Different Schemes,"* A thesis submitted to central Department of Mathematics, T.U., 2021.

5) Darae Jeong et al., *"Finite Difference Method for the Black–Scholes Equation Without Boundary Conditions,"* Computational Economics, 2017.

6) Hajipour M. and Malek A., *"Efficient High-order Numerical Methods for Pricing of Options,"* Springer, 2013.

7) Rouhparvar H. and Salamatbakhsh M, *"Analytical Solution of the Black-Scholes Equation by Variational Method"*, Applied Mathematics, 2013.

8) Merton Robert C., *"Theory of Rational Option Pricing,"* The Bell Journal of Economics and Management Science, Vol. 4, No. 1, 1973.

9) Ntwiga D.B., *"Numerical Methods for the Valuation of Financial Derivatives,"* University of Western Cape, 2005.

10) Nwozo C.R. and Faugba S.E., *"Some Numerical Methods for Options Valuation,"* Communications in Mathematical Finance, 2012.

11) Rao S. Chandra Shekhara and Manisha, *"High-Order Numerical Method for Generalised Black-Scholes Model,"* Elsevier, 2016.

12) Saedi Yasir Ahamad AI and Tularam Gurudeo Anand, *"A Review of the Recent Advances in the Black-Scholes Models and respective Solutions Methods,"* Journal of Mathematics and Statistics, 2017.

13) Sawangtong P., *"The Analytical Solution for the Black-Scholes Equation with the Two Assets in the Liouville-Caputo fractional Derivative Sense,"* ∑ Mathematics, 2018.

14) Smith G.D., *"Numerical Solution of Partial Differential Equations: Finite Difference Methods,"* Oxford University Press, 1985.

15) Tavella D. and Randall C, *"Pricing Financial Instruments the Finite Difference Methods*, John Wiley and Sons, Inc., 2000.

16) Wilmott Paul et al., *"Option Pricing: Mathematical Models and Computation"*, Oxford Financial Press, 1994.