SOME CLASSICAL SEQUENCE SPACES AND THEIR TOPOLOGICAL STRUCTURES

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Abstract

The aim of this paper is to study some of the basic scalar and vector valued sequence spaces .We also study the topological structures of some of the basic sequence spaces when topologized through a norm or a paranorm.

Keywords: *Function Space, Sequence Space, Paranormed Space, Orlicz Sequence Space.*

2010 AMS Subject Classification: Primary - 46A45, Secondary - 46B 15.

1. Introduction

Functional Analysis is an abstract branch of Mathematics, that deals with the study of linear spaces endowed with some kinds of limit-related structures like topology, norm, inner product etc. and the operators or functions acting upon these spaces.By a function space we mean a linear space of functions defined on a certain set with respect to pointwise addition and scalar multiplication.

The study of sequence spaces is in fact a special case of the more general study of function spaces if the domain is restricted to the set of natural numbers **N**.The set ω of all functions from the natural numbers **N** to the field **K** of real ***R*** or complex numbers ***C***, can be turned into a vector space. In other words,let ω be the set of all (real-or) complex valued sequences $\{x_n\}_{n \in N}$. i.e., $x_n \in \boldsymbol{C}$ under the operations of point wise addition and scalar multiplication given by

$$\{x_n\}_{n\in N} + \{y_n\}_{n\in N} = \{x_n + y_n\}_{n\in N}$$

$$\text{and } \lambda\{x_n\}_{n\in N} = \{\lambda x_n\}_{n\in N},$$

for every $x_n, y_n \in \boldsymbol{C}$ and scalar λ, form a vector space over **C**. Any subspace X of ω is then called a **sequence space**. In other words, a sequence space is a vector space whose elements are **infinite scalar** sequences of real or complex numbers and is closed under the coordinatewise addition and scalar multiplication. If it is closed under coordinate wise multiplication as well, it is called a **sequence algebra**. Sequence spaces when equipped with a linear topology form **topological vector spaces.**

Several workers like Kamthan and Gupta (1980), Maddox (1980), Ruckle (1981), Malkowski and Rakocevic (2004) etc. have made their significant contributions in developing the theory of vector and scalar valued sequence spaces in various directions, when sequences are taken from a Banach space or from a locally convex space. Literatures concerning the theory of sequence space can be found in any standard text books and monographs of Functional Analysis, for instance we refer a few; Lindenstrauss and Tzafriri (1977), Wilansky(1978), Kamthan and Gupta (1980), Rao and Ren (1991) etc.

2. Topological Structures of Some Basic Sequence Spaces

We shall study the topological structures of sequence spaces when topologized through a norm or through a paranorm generalize and unify various existing sequence spaces.The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value of real numbers or modulus of a complex number.

A paranormed space (X, G) is a linear space X together with a function $G : X \to \boldsymbol{R}_+$ (called a paranorm on X) which satisfies the following axioms:

$PN1$: $G(\theta) = 0$;

$PN2$: $G(x) = G(-x)$ for all $x \in X$;

$PN3$: $G(x + y) \leq G(x) + G(y)$ for all $x, y \in X$; and

$PN4$: if (α_n) be a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and (x_n) be a sequence in X with $G(x_n - x) \to 0$ as $n \to \infty$, then $G(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$ (continuity of scalar multiplication).

A paranormed space (X, G) is said to be *complete* if (X, d) is complete with metric

$$d(x, y) = G(x - y).$$

Every paranormed space becomes a linear pseudometric space with $d(x, y) = G(x - y)$ which is translation invariant. Thus each paranormed space is a topological linear space. Moreover a pseudometric linear space must be a paranormed space, i.e., a paranorm can be defined on the space which induces an equivalent pseudometric.

The studies on paranormed sequence spaces were initiated by Nakano and Simons at the initial stage. Later on it was further studied by Maddox [1969] and many others. Bhardwaj and Bala [2007], Khan [2008], and many others studied paranormed sequence spaces using Orlicz function.

We shall discuss the topological structures of some of the important classes of the basic sequence spaces when topologized through a norm or through a paranorm, which in fact, generalize and unify various existing sequence spaces. Among them, sequence spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ appear in the work of Lascarides and Maddox (1971) and others while $c_0(X)$, $c(X)$, $\ell_\infty(X)$, $\ell_p(X)$ are used by Leonard (1976),Maddox (1980) and others. Let $x = (x_k) = (x_k)_{k=1}^{\infty}$ be the sequences and ω denote the class of all sequences $x = (x_k)$, $k \geq 1$, over the field $\mathbf{C}$ of complex numbers. Let $p = (p_k)$ be any sequence of strictly positive real numbers (bounded in general) and $\lambda = (\lambda_k)$ be any sequence of non zero complex numbers. Here we deal with the topological structures of the following sequence spaces.

A. The spaces $c(p)$, $c_0(p)$, c and c_0.

With $\{p_k\}$ as above, define

$$c_0(p) = \{x = \{x_k\} : |x_k|^{p_k} \to 0 \text{ as } k \to \infty\};\text{and}$$

$$c(p) = \{x = \{x_k\}: |x_k - l\,|^{p_k} \to 0 \text{ as } k \to \infty \text{ for some } l \in \mathbf{C}\}.$$

$c(p)$ and $c_0\,(p)$ form metric spaces with the metric

$$d(x, y) = \sup_k |\,x_k - y_k|^{p_{k/M}}, \text{ where } M = max\ \{1, \sup p_k\}.$$

Note that the set $c(p)$ and $c_0(p)$ are complete with its metric topology. But it is not normed space. In fact, the set $c(p)$ forms a paranormed space with paranorm

$$\|\,x\,\| = \sup_k |x_k|^{p_{k/M}} \text{ if and only if } \inf p_k > 0.$$

The set $c_0\,(p)$ is a linear metric space paranormed by

$$\|\,x\,\| = \sup_k |x_k|^{p_{k/M}}, \text{ where } M = \max\ \{1, \sup p_k\}.$$

If $p_k = p$ for all k, then we write c and c_0 for $c(p)$ and $c_0\,(p)$ respectively. c and c_0 are respectively the sets of all convergent sequences and null sequences. Note that c and c_0 are metric spaces as well as Banach spaces with the metric and norm respectively given by

$$d(x, y) = \sup_k |x_k - y_k| \text{ and } \|\,\boldsymbol{x}\,\| = \sup_k |\,x_k\,|.$$

B. The spaces $l_\infty\,(p)$ and l_∞.

Let $\{p_k\}$ be as in the above. We define

$$l_\infty\,(p) = \{x = \{x_k\}: \sup_k |x_k|^{p_k} < \infty\}.$$

Note that $l_\infty\,(p)$ is complete metric space with its metric topology with the metric

$$d\,(x, y) = \sup_k |x_k - y_k|^{p_{k/M}}, \text{ where } M = max\ \{1, sup\ p_k\}.$$

But it is not normed space. In fact, the set $l_\infty\,(p)$ is paranormed space with paranorm

$$\|\,x\,\| = \sup |\,x_k|^{p_{k/M}} \text{ if and only if } \inf p_k > 0.$$

If $p_k = p$ for all k, then we write l_∞ for $l_\infty\,(p)$. l_∞ is the set of all bounded sequences $x = \{x_k\}$ of real (or complex) numbers and forms a metric space and Banach space with the natural metric and norm respectively given by

$$d(x,y) = \sup_k |x_k - y_k| \text{ and } \|\,x\,\| = \sup_k |x_k|.$$

C. The spaces $\ell\,(p)$ and ℓ_p.

Since $\{p_k\}$ be a bounded sequence of strictly positive real numbers, so that $0 < p_k \le \sup p_k < \infty$. We define

$$l\,(p) = \{x = \{x_k\}: \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}.$$

Then, $l\,(p)$ becomes a metric space with metric

$$d(x, y) = \left(\sum_{k=1}^{\infty} | x_k - y_k |^{p_k} \right)^{1/M}, \text{ where } M = \max \{1, \operatorname{Sup} p_k\}.$$

The space $l\,(p)$ is complete metric space in its metric topology but it is not a normed space.

If fact, the set $l(p)$ is paranormed by

$$\| x \| = \left(\sum_{k=1}^{\infty} | x_k |^{p_k} \right)^{1/M}.$$

In particular, if $p_k = p$ for all k, then we write l_p for $l\,(p)$, which consists of the p –power summable sequences with p-norm.

Observe that if $1 \le p < \infty$, l_p forms a metric space and Banach space with the metric and norm respectively given by

$$d(x, y) = \left(\sum_{k=1}^{\infty} | x_k - y_k |^{p} \right)^{1/p} \text{ and } \| x \| = \left(\sum_{k=1}^{\infty} | x_k |^{p} \right)^{1/p}.$$

Similarly for $0 < p < 1$, l_p forms a metric space and complete p-normed space with the metric and p-norm respectively given by

$$d(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|^p \text{ and } \| x \| = \sum_{k=1}^{\infty} |x_k|^{p}.$$

D. Orlicz Sequence Space ℓ_Φ

The study of Orlicz sequence spaces was initiated to study Banach space theory. An *Orlicz function* is a function $\Phi : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with

$\Phi(0) = 0$, $\Phi(u) > 0$ for $u > 0$, and $\Phi(u) \to \infty$ as $u \to \infty$.

Lindenstrauss and Tzafriri (1977) used Orlicz function to construct the sequence space

$$\ell_\Phi = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \Phi\left(\frac{| x_k |}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}$$

of scalars ,which forms a Banach space with Luxemburg norm defined by

$$\| x \|_\Phi = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{| x_k |}{\rho} \right) \le 1 \right\}.$$

The space ℓ_Φ is called an Orlicz sequence space and is closely related to the space ℓ_p with $\Phi(x) = x^p, (1 \le p < \infty)$. They have very rich topological and geometrical properties that do not occur in ordinary ℓ_p spaces.

Bhardwaj and Bala (2007), Khan (2008), Kolk (2011) ,Pahari and Srivastava (2011),(2012), and many others, have been introduced and studied various sequence spaces using Orlicz function as a generalization of well known sequence spaces .

E. Other Sequence Spaces

We shall denote e and $e^{(n)}$ ($n =$ 1,2, …) for the sequences such that $e_k = 1$ for $k = 1,2, \ldots$ and

$$e_k^{(n)} = \begin{cases} 1, & (\text{ for } k = n) \\ 0, & (\text{for } k \neq n). \end{cases}$$

Let m be any nonnegative integer, we denote the m-section of a sequence $x = \{x_k\}$ by $x^{[m]}$, i.e.

$$x^{[m]} = \sum_{k=1}^{\infty} . x_k e^{(k)}.$$

Further CS, l_1 and BS denote for the sets of all convergent, absolutely convergent and bounded series as described below:

The space of bounded series BS is the space of sequences X for which $\sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty.$

The space $BS,$ when equipped with the norm

$$\| x \|_{BS} = \sup_n \left| \sum_{k=1}^{n} x_k \right|,$$

forms a Banach space isometrically isomorphic to l^{∞}, via the linear mapping

$$(x_n)_{n\in N} \rightarrow \left(\sum_{k=1}^{n} x_k \right)_{n\in N}.$$

A sequence space X with a linear topology is called a ***K*-space** provided each of the maps $p_k : X \rightarrow$ ***C*** defined by $p_k(x) = x_k$, $x \in X$ is continuous for all $k = 1, 2, \ldots$

A K-space X is called an ***FK* - space** provided X is a complete linear metric space. A normed FK space is called a ***BK*-space**.

Further , l_1 is a BK space with respect to the norm

$$\| x \|_1 = \sum_{k=1}^{\infty} | x_k |,$$

c_0, c and l_∞ are BK spaces with respect to the norm

$$\| x \|_\infty = \sup_k | x_k |,$$

and cs is a BK space with respect to the norm

$$\| x \|_{CS} = \sup_n \left| \sum_{k=1}^{n} . x_k \right|,$$ see, Malkowski (1999).

A *FK* space $X \supset \phi$ is said to have ***AK- space*** if every sequence $x = \{x_k\}_{k=1}^{\infty}$ in X has a unique representation of the form $x = \sum_{k=1}^{\infty} x_k e^{(k)}$, i.e. $x^{[m]} \to x$ (as $m \to \infty$).For examples, the spaces l_1, c_0 and *CS* form *AK* spaces.

A sequence $(x^{(k)})_{k=1}^{\infty}$ in a linear metric space X is called a ***Schauder basis*** (see, Malkowski ,1999), if for every $x \in X$ there exists a uniquely determined sequence $\{\lambda_k\}_{k=1}^{\infty}$ of scalars such that

$$x = \sum_{k=1}^{\infty} \lambda_k x^{(k)}.$$

A complex sequence $\{x_k\}$ is called an ***analytic sequence*** if the sequence $\{|x_k|^{1/k}\}$ is bounded and an ***entire sequence*** if $|x_k|^{1/k} \to 0$ as $k \to \infty$. The set of all analytic and entire sequences are respectively denoted by ϖ and Γ, see Rao (1999).

3. Conclusion

In this paper, we have characterized the topological structures of some of the basic scalar and vector valued sequence spaces. In fact, these structures can also be used to explore further properties of the generalized sequence spaces by topologizing through a norm or a paranorm.

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