Summability, Some Sequence, Some Sequence Spaces and Their Matrix Transformations

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ABSTRACT
The most general linear operator to transform from new sequence space into another sequence space is actually given by an infinite matrix. In the present paper we represent some sequence spaces and their matrix transformations and summability.

KEY WORDS
Duals, Kothe- Toeplitz , matrix transformation Sequence space, summability.

INTRODUCTION

Concepts of summability:

Let A =(a_{nk}) $\infty_{n,\ k=1}$ be an infinite matrix, x = (x_k) $\infty_{n,\ k=1}$ be a sequence, e =1, 1, 1 · · ·), A_nx =$\sum_{k=1}^{\infty} a_{nk}x_k$ and Ax = A =(a_nx) $\infty_{n,\ k=1}$ be the sequence ofthe A transforms of x. There are three concepts of summability.

• Ordinary summability : x is summable A if

limn→∞ A_nx = ℓ for some ℓ ∈ |C

• strong summability : x is strongly summable A with index p > 0 if

limn→∞A_n(|x − ℓ · e|^p) = limn→∞$\sum_{k=1}^{\infty} a_{nk}$ |xk − ℓ|p = 0

ank|xk − ℓ|^p = 0 for some ℓ ∈ |C

• absolute summability : x is absolutely summable A with index p > 0 if

$\sum_{k=1}^{\infty}$ |A_nx − A_n − 1x|p < ∞.

An example

Example 1.1 Let the matrix A be given by a_{nk} = 1/n for 1 ≤ k ≤ n and

a_{nk} = 0 for k > n (n = 1, 2, . . .). Then the A transforms of the sequence

x are the arithmetic means of the terms of x, that is,

σ_n =1/n$\sum_{k=1}^{\infty} x_k$ and A defines the Ces`aro method C1 of order 1.

• Every convergent sequence is summable C1 and the limit is preserved

• the divergent sequence $((-1)^k)^k_{k=1}$ is summable C1 to 0

• strong summability of index 1 implies ordinary summability to the same

limit; the converse is not true, in general

• absolute summability with index 1 implies ordinary summability

A sequence space is a linear space of functions defined on the set of counting numbers. Thus the sequence space is set of scalar sequence (real or complex) which is closed under coordinate wise addition and scalar multiplication. If it is closed under co-ordinate wise multiplication as well, then it is called the sequence algebra. We are concerned mainly on the problem of identification, inclusion problem and matrix mapping problems. The study of sequence spaces is thus a special case of the more general study of function space, which is in turn a branch of functional analysis.

Here, we begin some definitions and notations:

Normed Space:

Normed Space is a pair (X, $||.||$) of a linear space X and norm ||.|| on X.

Banach Space:

A Banach Space (X, || ||) is a complete normed space where completeness means that every sequence (x_n) in X with $\| x_m - x_n\| \to 0$ as m, n→∞, there exists x εX

such that $\|x_n$-x$\|$ →0 as n→∞.

Paranorm:

A paranorm 'g' defined on a linear space X, is a function: X→ R having the following usual properties:

(i) g(θ)= o, where θ is the o element in X.

(ii) g(x) = g(-x), for all x $\varepsilon X.$

(iii) g(x + y) ≤ g(x) +g(y) for all x, y $\varepsilon X.$

(iv) The scalar multiplication is continuous that is $\lambda_n \to$ λ (n→ ∞) and g(x_n-x) → o as n→ θ, for λ_n λ ε C and x_n x ε X, g($\lambda_n x_n$- λ x) → o as n → ∞.

(v) g(x) =o → $x = o.$

A paranormed space:

A paranormed space is a linear space X together with a paranorm g.

The space l_∞(p) :

Let $\{p_k\}$ be abounded sequence of strictly positive real numbers. We define

$l_\infty(p)$= { x= $\{x_k\}$: $\sup_k |x_k|^{p_k} < \infty$}

For x , y ε $l_\infty(p)$,we define

d(x, y)= $\sup_k | x_k - y_k |^{p_k/M}$

Where M = max (1, sup p_k). $l_\infty(p)$ is a metric space with metric d.

If p_k =p for all k, then we write $l_\infty$for $l_\infty(p)$. Here l_∞ is the set of all bounded sequences x = $\{x_k\}$ of real or complex numbers and is a metric space with the natural metric

d(x, y) = $\sup_k |x_k - y_k|$.

Spaces c(p) and c_o(p) :

With $\{p_k\}$, we define

c(p) ={x = $\{x_k\}$: $|x_k - l|^{p_k}$→0 as k→∞ for some l ε C} and

c_0(p) ={x = $\{x_k\}$: $|x_k|^{p_k}$→0 as k→∞ }

c(p) and c_0(p) are the metric spaces with metric

d(x , y) = $\sup_k | x_k - y_k |^{p_k/M}$, where M = max (1, sup$p_k$).

The spaces c and c_o:

If p_k = p for all k, then we write c and c_0 for c(p) and c_0(p) respectively. c and c_0 represent the sets of all convergent sequences and null sequences respectively.

Note that c and c_0 are metric spaces with the metric

d(x ,y) = $\sup_k |x_k - y_k|$.

In c if we define p(x, y) = | lim($x_n - y_n$) |,

then although p(x, y) =0, this does not always imply that x = y.

For example if we take $\boldsymbol{x_k}$ =1/k and $\boldsymbol{y_k}$=0 for all k, observe that the other two axioms of a metric are satisfied by $\boldsymbol{\rho}$Thus $\boldsymbol{\rho}$ is not a metric on c, but is a semi metric.

Duals:

If X is a sequence space, We define

x^{β} = {a=(a_k) : $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each x ε X}.

Theorem (1):

Let $p_k > o$ for every k, then

$[Sl_\infty(p)]^\beta$ =$\bigcap_{N=2}^{\infty}\{ a = \{a_k\}$:$\sum_{k=1}^{\infty} a_k[\sum_{m=1}^{k} N^{1/p_m}$]} converges$\sum_{k=1}^{\infty} N1/p_k$ $|R_k| < \infty$, $N > 1$, where $R_k = \sum_{v=k}^{\infty} a_v$ (we assume that $\sum_{m=1}^{k} z_m$ = o (k > 1)).

Proof: Suppose that x $\varepsilon\ SL_\infty(p)$, we choose N > 1, so that sup$p_{k|\Delta x_k|}p_k < N$,we write

$\sum_{k=1}^{m} a_k x_k = \sum_{k=1}^{m} R_k \Delta x_k - R_{m+1} \sum_{k=1}^{m} \Delta x_k$ $(m = 1, 2, 3, \ldots)$ (1)

Since $\sum_{k=1}^{\infty} |R_k|\ |\Delta x_k| \leq \sum_{k=1}^{\infty} |R_k|\, N1/P_k < \infty$, *it follows that* $\sum_{k=1}^{\infty} R_k \Delta x_k$ is absolutely convergent. By corollary 2 in [6], the convergence of $\sum_{k=1}^{\infty} a_k$($\sum_{m=1}^{k} N$1/Pm) implies that $\lim_{m\to\infty} R_{m+1\sum_{m=1}^{k} N}$1/Pm = o. Hence, it follows from (1) that$\sum_{k=1}^{\infty} a_k x_k$ is convergent for each x $\varepsilon\ Sl_\infty$ (p). This yields a $\varepsilon\ (Sl_\infty(p))^\beta$.

Conversely, suppose that a$\varepsilon\ (Sl_\infty(p))^\beta$, then by definition, $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each x $\varepsilon\ SL_\infty(p)$.

Since e = (1, 1, 1,..) $\varepsilon\ Sl_\infty(p)$and x = [$\sum_{m=1}^{k} N$1/Pm] $\varepsilon\ Sl_\infty(p)$ so,

$\sum_{v=1}^{\infty} a_v$ and $\sum_{v=1}^{\infty} a_v$[$\sum_{m=1}^{v} N$1/Pm] are respectively convergent. By using corollary 2 in [2o], we find that

$\lim_\infty R_{m+1} \sum_{m=1}^{v} N$1/Pm = o.

Thus, we get from (1) that the series $\sum_{k=1}^{\infty} R_k \Delta x_k$ converges for each x$\varepsilon\ Sl_\infty(p)$.

Since x $\varepsilon\ Sl_\infty(p)$ if and only if $\Delta x\ \varepsilon\ Sl_\infty(p)$. *This implies that* $R = \{R_k\ \varepsilon(\ Sl_\infty(p))^\beta$. It now follows from a theorem 2 in [1o] that $\sum_{k=1}^{\infty} |R_k|\ N$1/pk converges for all N > 1.

This completes the proof of the theorem.

Theorem (2):

Let $p_k > o$, *for every k, then*

$[Sc_o(p)]^\beta$ = SM_o(p), where SM_o(p) = $\bigcup_{N>1}\{a = \{a_k\}$: $\sum_{k=1}^{\infty} a_k$ [$\sum_{m=1}^{k} N$-1/Pm] converges and $\sum_{k=1}^{\infty} |R_k|\ N^{-1/p_k} < N > 1\}$.

Proof:

Let a ε SM_o(p) and x ε Sco(p). We choose an integer N> 1 $such\ that$ $|\Delta x_k|$pk $<$ N-1.

We have $\sum_{k=1}^{m} a_k x_k$ = $\sum_{k=1}^{m} R_k \Delta x_k$ - $R_{m+1} \sum_{k=1}^{m} \Delta x_k$; (m = 1, 2, 3, …).

Since $\sum_{k=1}^{\infty} |R_k \Delta x_k| \leq \sum_{k=1}^{\infty} |R_k||\Delta x_k| \leq \sum_{k=1}^{\infty} |R_k|\ N^{-1/p_k} < \infty$, it follows that,

$\sum_{k=1}^{\infty} R_k \Delta x_k$ is convergent absolutely. The convergence of

$\sum_{k=1}^{\infty} a_k$ ($\sum_{m=1}^{k} N$-1/Pm) implies that

$R_{m+1} \sum_{k=1}^{m} N$-1/Pi = o (1) (m→ ∞). Hence $\sum_{k=1}^{\infty} a_k x_k$ converges for each x ε SM_o(p). That is,

a ε [Sc_o(p)]$^{\beta}$.

Conversely, let a ε[Sc_o(p)]$^{\beta}$, $then$

$for\ any$ x ε SM_o(p), $\sum_{k=1}^{\infty} a_k x_k$ converges. Since the sequence x={$\sum_{m=1}^{k} N$-1/Pm} by choosing $\epsilon > \frac{1}{N}, (N = 2, 3, \dots)\varepsilon$ Sc_o(p) it follows that $\sum_{k=1}^{\infty} a_k$

($\sum_{m=1}^{k} N$-1/Pm) converges [Because $\sum_{m=1}^{k} N$ -1/pm ε Sco(p)]

To show that $\sum_{k=1}^{\infty} |R_k|\ N^{-1/p_k} < \infty,\ N > 1$, let us assume that $\sum_{k=1}^{\infty} |R_k|\, N^{-1/p_k} < \infty,\ N > 1$, then from Theorem 6 , it follows that R $\notin$ Mo(p) = [$c_o(p)$]$^{\beta}$, then there exists a sequence x = {1/k}, k ≥ 1 ε c_o(p) such that

$\sum_{k=1}^{\infty} R_k$ 1/k does not converse. Although, if we define

y = {y_k } by y_k = $\sum_{n=1}^{k} \frac{1}{n}$, then, y ε Sco(p), but $\sum_{k=1}^{\infty} a_k y_k$ = $\sum_{k=1}^{\infty} a_k${ $\sum_{n=1}^{k} \frac{1}{n}$ } = $\sum_{k=1}^{\infty} R_k$ 1/k.

Hence $\sum_{k=1}^{\infty} a_k y_k$ does not converge for y ε Sco(p), a contradiction is due to the fact that

a ε [Sc_o(p)]$^{\beta}$. So

$\sum_{k=1}^{\infty} |R_k| N^{-1/p_k} < \infty,\ N > 1.$

This completes the proof of the theorem.

MATRIX MAPS:

Let X and Y be any two sequence spaces. Let A = ($a_{n,k}$)$_{n,k=1}^{\infty}$

($1 \leq n,\ k\ \leq\ \infty$) be an infinite matrix of scalar entries.

$Ax = \left(A_n\ (x)\right)_{n=1}^{\infty} \varepsilon\, Y$, where A_n(x) = $\sum_{k=1}^{\infty} a_{nk} x_k$ is a convergent sequence for each n (n = 1, 2, 3,…). We say that A defines a matrix map from X into Y and we write A ε (X, Y). By (X, Y), we mean the class of matrices A such that A ε (X, Y). $The\ main\ aim\ is\ to$ characterize the spaces (Sl_{∞}(p), c_s). We shall first establish the following simple lemma 1.

Lemma (1):

Let X and Y be two sequence spaces, and let$\Delta Y =$ {y ={y_k }: $\Delta y = (y_k - y_{k-1})$ ε Y, y_o= o}, then A ε (X, Y) if and only $\Delta A = (a_{n,k} - a_{n-1,k})_{n,k=1}^{\infty}$ =($b_{n,k}$)$_{n,k=1}^{\infty}$ = B ε (X, Y).With lemma1,. (i, ii) in [10] or, Theorem 3 in [10] or, Theorem 5b (i) and Theorem7 in [24], a characterization of the classes ($l(p), Sl_{\infty}$) or ($l_{\infty}(p),\ Sl_{\infty}$) or (($l(p),\ Sl_{\infty}(q)$) (q ε l_{∞}) immediately follows

In [6] the authors have characterized the spaces (Sl_{∞}(p), l_{∞}) iff the matrix A satisfy following the conditions:

Theorem3:

Let $p_k > o$ for every k then, A ε (Sl_∞(p), l_∞) if

(i)$\sup_n | \sum_{k=1}^{\infty} a_{nk}(\sum_{m=1}^{k} N^{1/p_m})]| < \infty, N > 1.$

(ii) $\sup_n [\sum_{k=1}^{\infty} N^{1/p_k} |\sum_{v=k}^{\infty} a_{nv}| < \infty, N > 1.$

Proof: We first prove that these conditions are necessary.

Suppose that Aϵ $(sl_\infty$(p),l_∞). Since x= (x_k) = ($\sum_m^k N^{1/p_m}$)

belongs to sl_∞(p), the condition (i)holds. In order to see that (ii) is necessary we assume that for N>1,

$$sup_n[\sum_{k=1}^{\infty} N^{\frac{1}{p_k}} |\sum_{v=k}^{\infty} a_{nv}|] =\infty.$$

Let the matrix B be defined by

$$B = (b_{nk}) = (\sum_{v=k}^{\infty} a_{nv}).$$

Then it follows from Theorem 1.12.8 that B $\notin$ $(sl_\infty$(p),l_∞). Hence, there is a sequence x ϵ sl_∞(p) such that

$$sup_k |x_k|^{p_k} = 1 \text{ and } \sum_{k=1}^{\infty} b_{nk}x_k \neq O(1).$$

We now define the sequence y = (y_k) by

$$y_k = \sum_{v=1}^{k} x_v \ \ (k \in \mathbb{N}),$$

$$y_o = o.$$

Then y $\in$ sl_∞(p) and $\sum_{k=1}^{\infty} a_{nk}y_k = \sum_{k=1}^{\infty} b_{nk}x_k \neq O(1).$

This contradicts that Aϵ $(sl_\infty$(p),l_∞). Thus, (ii) is necessary.

We now prove the sufficiency part of the theorem.

Suppose that (i) and (ii) of the theorem hold. Then $A_n \in (sl_\infty(\text{p}))^\beta$ for each n $\in$ $\mathbb{N}$.

Hence $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for each n $\in$ $\mathbb{N}$ *and for each* x ϵ sl_∞(p). Following the argument used in lemma 1, we find that if x ϵ sl_∞(p) such that $sup_k | \Delta x_k|^{p_k}$ < N, then

$$|\sum_{k=1}^{\infty} a_{nk}x_k| \leq \sum_{k=1}^{\infty} N^{\frac{1}{p_k}} |\sum_{v=k}^{\infty} a_{nv}| ;$$

$$\leq sup_n[\sum_{k=1}^{\infty} N^{\frac{1}{p_k}} |\sum_{v=k}^{\infty} a_{nv}|];$$

$$< \infty.$$

This proves that AX $\in$ l_∞. Hence, the theorem is proved.

Theorem (4):

Let $p_k >$ o, for every k, then A ε ($Sl_\infty(p)$, c) if and only if

(i)R ε ($l_\infty(p)$, c) where R = $(r_{n,k}$) = [$\sum_{v=k}^{\infty} a_{n,v}$] (n, k = 1, 2, 3,…).

(ii) A_n [$\sum_{i=1}^{k} N^{\frac{1}{p}}$ i] ε c (n, k =1, 2, 3,…) for all integers, N $> 1.$

(iii) $\lim_{n\to\infty} a_{n,k}\ \alpha_k$ (k = 1, 2, 3,…).

Proof: Let us first prove the sufficiency condition. For consider any x ε $Sl_\infty(p)$, we choose N $>$ $1,$ *so that* $supp_k |\Delta x_k|^{p_k} < N.$ *w*e write,

$\sum_{k=1}^{n} a_{n,k}x_k = \sum_{k=1}^{m} a_{n,k}\Delta x_k - r_{n+1,m}\sum_{k=1}^{m}\Delta x_k$ (m = 1, 2, 3,…). (2).

By condition (ii) $\sum_{k=1}^{\infty} a_{n,k}[\sum_{i=1}^{k} N^{\frac{1}{p}}{}_i]$ is convergent for each (n = 1, 2, 3,…).Hence, by corollary 2 in [20] it follows that

$\lim_{m\to\infty} r_{n+1,m} \sum_{i=1}^{k} N^{\frac{1}{p}} i$ = o. By condition (i), R $\varepsilon\,(l_\infty(p), c)$, and since x ε S$l_\infty(p)$ *if and only if* $\Delta x\ \varepsilon\ l_\infty(p)$. *Hence, by corollary* [2] *in* [20] *it follows that*

$\sum_{k=1}^{\infty}|r_{n,k}|N^{1/p_k}$ is uniformly convergent in n and $\lim_{n\to\infty} r_{n,k}$ exists for each (k = 1, 2, 3,…)

Since $\sum_{k=1}^{\infty}|r_{n,k}||\Delta x_k| \le \sum_{k=1}^{\infty}|r_{n,k}| N^{1/p_k}$, from (2) we find that $\sum_{k=1}^{\infty} a_{n,k}x_k$ is absolutely and uniformly convergent in n. Finally, we have

$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}x_k = \sum_{k=1}^{\infty}\alpha_k x_k$. This proves the sufficiency condition.

The necessities of (iii) and (ii) are respectively obtained by taking x = e = (1, 1, 1,…) ε S$l_\infty(p)$ *and* x = $[\sum_{i=1}^{k} N^{\frac{1}{p}}{}_i]$ (k = 1, 2, 3,…), i ε S$l_\infty(p)$.Now consider the necessity of (i).If it is not true, then there exists x = $(x_v)\ \varepsilon\ l_\infty(p)$ with supp$_v|x_v|p_v$ = 1 such that $[\sum r_{n,v}x_v]^\infty \notin c$. *Al*hough if we define a sequence y = (y_k) by

$y_v = \sum_{i=1}^{v} x_i$ (v = 1, 2, 3,…), then y ε S$l_\infty(p)$ but $[\sum_{v=1}^{\infty} a_{n,v}y_v = \sum_{v=1}^{\infty} r_{n,v}x_v] \notin c$. This contradicts the fact that A ε (S$l_\infty(p)$, c) and therefore (i) must hold.

Before characterizing the class (S$l_\infty(p)$, c_s), we add one more notation, for any

n > 1, *we write*

t_n (AX) = $\sum_{t=1}^{v} A_i\ (x) = \sum_{k=1}^{\infty} b_{n,k}x_k$, [x ε S$l_\infty(p)$], whereB = $(b_{n,k})$ = [$\sum_{i=1}^{n} a_{i,k}$]

(n = 1, 2, 3, …).This complete the proof of the theorem.

Theorem (5):

Let $p_k >$ o, for every k, then A ε (S$l_\infty(p)$, c_s) if and only if

(i)C ε ($Sl_\infty(p)$, c_s) where C = $(C_{n,k})$ = $\{\sum_{i=1}^{n}[\sum_{V=K}^{\infty} a_{rv}]\}$ (n, k = 1, 2, 3,…).

(ii) $B_n\ [\sum_{i=1}^{k} N^{\frac{1}{p}}]\ \varepsilon\ c_s$ (n, k =1, 2, 3,…) for all integers, N > 1.

(iii) $\lim_{n\to\infty} b_{n,k} = \lim_{n\to\infty}\sum_{i=1}^{n} a_{i,k} = \beta_k$ (k = 1, 2, 3,…).

Proof:

This theorem follows immediately from theorem (4);

Let us first prove the sufficiency condition. For consider any x ε S$l_\infty(p)$, we choose N > 1, *so that* suppk $|\Delta x_k|$pk $< N$. *w*e write,

$\sum_{k=1}^{m} b_{n,k}x_k = \sum_{k=1}^{m} c_{n,k}\Delta x_k - C_n$, m+1 $\sum_{k=1}^{m}\Delta x_k$ (m = 1, 2, 3,…) and the convergence of

$\sum_{k=1}^{\infty} b_{n,k}[\sum_{i=1}^{m} N^{1/p_i}]$ implies that

$\lim_{m\to\infty} C_{n,m+}\ 1\sum_{i=1}^{m} N^{1/p_i}$ = 0.

Characterization of $(l(p), Sc_o(q))$, qεl_∞ follows from Theorem 5 (ii) [28] with lemma 1.

This completes the proof of the theorem.

CONCLUSION

The results obtained in this research paper are very closely linked with the summability theory and matrix transformations. So the practical applications of this research paper have the same applicability applications as those of summability theory and matrix transformation between sequence spaces.

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