

Interval and Speed of Convergence on Iterative Methods

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Abstract : The iterative method is a tool of solving the non-linear equations to get their approximate roots with some errors of tolerance. Repetition of the similar process is applied successively on such iterations to compute a sequence of increasingly accurate estimates of the roots. In this paper, the construction of an iterative method for solving an equation, its convergence and the determination of interval of convergence for the approximate choice of initial guess and the speed of convergence are highlighted.

Keywords: Iteration, convergence, convergence interval, speed of convergence

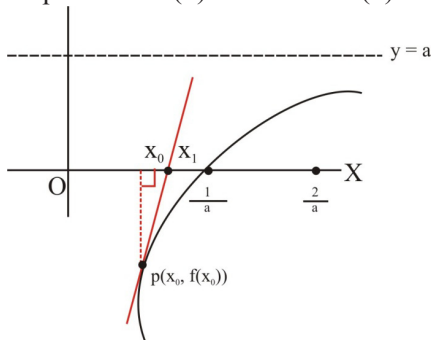
1. INTRODUCTION

The rate at which a convergent sequence approaches its limit is the speed of convergence. The set of all values of x for which the sequence converges gives the interval of convergence. A power series often converge or diverge, depending upon the values of x . If converges, the nature of convergence is also different.

Finding one or more roots of an equation $f(x) = 0$ -----
 --- (1) is one of the most commonly occurring problems of applied mathematics, specially on the numerical methods. For such equations, the explicit solutions are not available and their better approximations are obtained by different iterative methods. An iterative method is a powerful device of solving and finding the roots of the non-linear equations. Iteration means the act of repeating the similar process to achieve the desired goal where the result of one iteration is used as the starting point for the next operation. Successive approximations are used in each step to get more accurate solutions to a linear system and it involves a large number of iterations of arithmetic operations.

2. CONVERGENCE OF X_N

Consider the equation be $f(x) = a - x^2 = 0$ -----(2) for given $a > 0$.



Let x_0 be the initial guess. Let the tangent at $p(x_0, f(x_0))$ meets the x -axis as x_1 , then it becomes the better approximation of the root than x_0 .

Slope of the tangent line =

$$f'(x_0) = \dots\dots\dots (3)$$

Then, (2) becomes

$$+ =$$

$$\text{or, } x_0 - x_1 = ax_0^2 - x_0$$

$$\text{or, } x_1 = 2x_0 - ax_0^2.$$

$$\therefore x_1 = x_0(2 - ax_0) \dots\dots\dots (4)$$

Equivalent to say, $x_{n+1} = x_n(2 - ax_n) \dots\dots\dots (5)$ for $n \geq 0$, for the iterative methods.

$$\text{Let } r_n = 1 - ax_n \dots\dots\dots (6) \text{ then } x_{n+1} = x_n(1 + r_n)$$

where the error be

$$e_n = x_n - \dots\dots\dots$$

$$\therefore e_n = \dots\dots\dots (7)$$

From (6), we can write

$$r_{n+1} = 1 - ax_{n+1}$$

$$= 1 - ax_n(1 + r_n), \text{ from (5)}$$

$$= 1 - (1 - r_n)(1 + r_n), \text{ from (6)}$$

$$= 1 - 1 + r_n^2$$

$$\therefore r_{n+1} = r_n^2 \dots\dots\dots (8)$$

Here,

$$n = 0 \Rightarrow r_1 = r_0^2$$

$$n = 1 \Rightarrow r_2 = r_1^2 = (r_0^2)^2 = r_0^{2^2}$$

$$n = 2 \Rightarrow r_3 = r_2^2 = (r_0^4)^2 = r_0^{2^3}, \text{ and so on.}$$

$$\therefore r_n = r_0^{2^n} \dots\dots\dots (9) \text{ for } n \geq 0.$$

Thus from (9), we can say,

$$r_n \rightarrow 0 \text{ iff } |r_0| < 1$$

$$\text{iff } -1 < r_0 < 1$$

iff $-1 < 1 - ax_0 < 1$, from (6)

iff $-2 < -ax_0 < 0$

iff $0 < ax_0 < 2$

iff $0 < x_0 < \dots$ (10) for $a > 0$.

Hence, for the convergence of x_n to α , it is necessary and sufficient that x_0 be chosen in between 0 and $\frac{2}{a}$; which tells us the interval of convergence on different iterative methods.

Speed of Convergence

The speed of convergence of a sequence can be determined as follows:

Let the sequence of real numbers be x_0, x_1, x_2, \dots converges to α . The speed of convergence is the rate at which the numbers are converging to α .

If, $\lim_{n \rightarrow \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = \lambda < \infty$, then the sequence converges linearly to α where the constant λ is called the asymptotic error.

$$\text{If, } \lim_{n \rightarrow \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^2} = \lambda < \infty,$$

then the sequence converges quadratically to α .

$$\text{Thus, if } \lim_{n \rightarrow \infty} \frac{|x_{n+1} - \alpha|^p}{|x_n - \alpha|^p} = \lambda < \infty,$$

then the sequence converges to α of order p .

Examples:

i) Let $x_n = \frac{1}{(n+1)^k}$ for some $k > 0$.

Clearly, it converges to 0, where the rate of convergence be given by

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^{ak}}} = \lim_{n \rightarrow \infty} \left(\frac{n^\alpha}{n+1} \right)^k$$

We get the convergence for $\alpha = 1$, but not for $\alpha = 2$.

Hence, the above sequence converges linearly to 0.

In fact, $x_n = \frac{1}{n}, x_n = \frac{1}{n^2}, x_n = \frac{1}{n^3}, \dots$ all converge linearly to zero.

ii) Let $x_n = \frac{1}{2^{2^n}}$, which also converges to 0.

For the rate of convergence,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n \rightarrow \infty} \frac{2^{-2^{n+1}}}{2^{-2^n} \cdot 2} = \frac{1}{2}$$

\ we get the convergence for $\alpha = 2$.

Thus for the terms of the sequence be like $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, 10^{-32}, 10^{-64}, \dots$ which shows the very fast convergence of order 2, quadratically convergent.

Relative error

The magnitude of the difference between the exact value and the approximation is absolute error where the relative error is simply the ratio of absolute error to the exact value which can also be expressed in terms of percentage.

To examine the rate of decrease on the relative error, let us examine the speed of convergence from (6).

For this we write the above mentioned result be,,

$$e_{n+1} = e_n^2 = a e_n^2$$

$$\text{or, } e_{n+1} = e_n^2 a^2 =$$

$\rho \text{ Rel}(x_{n+1}) = [\text{Rel}(x_n)]^2$, for $n \geq 0$, where $\text{Rel}(x_n)$ denotes the relative error in x_n .

So, to know how fast the relative error will decrease, let $\text{Rel}(x_0) = 0.1$, where,

$$\text{Rel}(x_1) = [\text{Rel}(x_0)]^2 = 0.01 = 10^{-2}$$

$$\text{Rel}(x_2) = [\text{Rel}(x_1)]^2 = 0.0001 = 10^{-4}$$

$$\text{Rel}(x_3) = [\text{Rel}(x_2)]^2 = 10^{-8}$$

$$\text{Rel}(x_4) = [\text{Rel}(x_3)]^2 = 10^{-16} \dots \text{ and so on.}$$

Here, the relative errors in each iterations double the number of significant digits, which provides the nature of the rate of its decrease in the relative errors.

3. CONCLUSION

An equation can be solved with the better approximations by the iterative approach with some reasonable errors. For the solution procedure, iterative method is a better tool for different non-linear equations to get their approximate solution where the repetition of the similar process is carried out successively to compute a sequence of increasingly accurate estimation of their solutions. In the convergence analysis of the solution of non-linear equations, the initial guess or the initial approximation within the interval of convergence and the speed of convergence play the vital role.

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