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Optimization Problem in Economics: The Classical Method and Inequality Constraint

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INTRODUCTION

The meaning of optimization is to find the best desirable point. The problem of optimization may be interpreted in two ways. First interpretation refers to a broad area of maximization and minimization problems irrespective of any constraint. The second interpretation refers only to the constraint optimization. Although writers also talk about unconstraint optimization, most of the problems we deal with are constraint optimization in Economics. The problem of constraint optimization involves maximization or minimization of some objective function with restrictions on choice variables. We may have more than one constraint and, also, constraints may be in inequality form rather than in equality form. Any problem involving twice differentiable objective function with n choice variables and m constraints $(n>m)$ can be easily tackled by classical method using differential calculus.

The present paper has been designed to work out how the classical method may be modified so as to tackle the problems involving inequality constraints. In this connection important terminologies and the popular classical technique are reviewed. And required modifications are worked out to tackle the problems involving inequality constraints.

EQUILIBRIUM STATE AND OPTIMUM POINT

The optimum position of any economic unit is also the equilibrium position. However, it is not necessary that all the equilibrium positions should be optimum points. To comprehend it clearly we must have idea regarding the goal equilibrium and nongoal equilibrium. Let us first define the term equilibrium. According to one definition, an equilibrium is "a constellation of selected interrelated variables so adjusted to one another that no inherent tendency to change prevails in the model which they constitute" (Machlup; 1958: 9). This definition relates the term equilibrium with the state of rest owing to the balance of internal forces. We do not observe any term in this definition which relates equilibrium state to the optimum or best desirable state. Thus, we say that all the equilibrium points need not necessarily be the optimum points. The reason is that some of the equilibrium states may not be desirable, just like less than full employment level of income.

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The equilibrium positions may be broken down between goal and non-goal equilibrium. The goal equilibrium is defined as "the optimum position for a given economic unit (a household, a business firm or even an entire economy) and in which the said economic unit will be deliberately striving for attainment of that equilibrium" (Chang: 1984, 232). The nongoal equilibrium is antithesis of this which results "not from any conscious aiming at a particular objective but from an impersonal or suprapersonal process of interaction and adjustment of economic forces" (Chang: 1984, 36). If both market forces demand and supply are let free to attain any equilibrium point, that is called nongoal equilibrium. If there is government's effort to peg the price, it may be called goal equilibrium. A goal equilibrium is always desirable, but a nongoal equilibrium may or may not be desirable.

METHODS OF OPTIMIZATION

Over last 150 years differential calculus has been the major technique of optimization in physical science, geometry and engineering. In economics also it has been usefully applied in the theory of production and consumption. But this classical technique of optimization is inapplicable in some cases. Accordingly the quest for applicable solution has led to new development, that is, mathematical programming.

Classical Method

As has been said already, the classical optimizing technique involves the use of differential calculus. Sometimes economists talk about the case of unconstraint optimization, in which the optimizing technique is same as the technique for a simple problem of maxima or minima of calculus. Just what is required is the twice differentiable continuous objective function. But practically an economist hardly come across unconstraint optimization because of the fact that the economics is the science of choice, caused by scarcity of resources. The common optimizing problems we most frequently encounter in Economics are utility maximization subject to the budget constraint, profit maximization subject to the resource constraint, budget minimization subject to the utility constraint and the cost minimization subject to the output constraint. There can be one or more than one constraint as well, all of which are easily talked through the Lagrangian method, provided that constraints are in equality form.

The Lagrangian method is reviewed below:

In Economics most of the optimum points are defined at a tangency between the level curve of the objective function and the constraint curve. To give a mathematical treatment to this, let the objective function and the budget constraint for two variables case be

$G(x_1,x_2) = c$,   and

$F(x_1,x_2) = 0$ respectively.

If $\bar{x}$ is the optimum point, their slopes at $\bar{x}$ will be equal, i.e.:

$$\frac{G_1(\bar{x})}{G_2(\bar{x})} = \frac{F_1(\bar{x})}{F_2(\bar{x})} \quad \ldots\ldots \quad (1)$$

This equality of slopes does not guarantee maximization or minimization, because the same necessary condition results in both cases. Thus there is need for the technique which fulfils this necessary condition, guarantees maximization or minimization and still statisfies the constraint. It is possible to get such a technique with the help of equation (1). Writing equation (1) in another way:

$$\frac{G_1(\bar{x})}{F_1(\bar{x})} = \frac{G_2(\bar{x})}{F_2(\bar{x})} \quad \ldots\ldots \quad (2)$$

Supposing this to be equal to some constant λ:

$$\frac{G_1(\bar{x})}{F_1(\bar{x})} = \frac{G_2(\bar{x})}{F_2(\bar{x})} = \lambda$$

We obtain

$$\left.\begin{array}{l} G_1(\bar{x}) - \lambda\, F_1(\bar{x}) = 0 \\ G_2(\bar{x}) - \lambda\, F_2(\bar{x}) = 0 \end{array}\right\} \ldots\ldots \quad (3)$$

From equations (3) we can define a new function as

$$L(x,\lambda) = G(x) - \lambda\, F(x) \quad \ldots\ldots \quad (4)$$

This equation (4) produces equation (3) as its partial derivatives. It is well known calculus result that if a function is being maximized or minimized its first order derivatives are set equal to zero. This equation (4) is called Lagrangian function and is an alternative method for optimization, known as Lagrangian method. Like any maximization or minimization problem, optimization through Lagrangian method also involves first order and second order conditions. The first order conditions are to set the partial derivatives of the Lagrangian function equal to zero, but the second order conditions vary according to the nature of the optimization problem and the number of constraints.

One Constraint Case

Let the objective function and the constraint be $f(x_1,x_2\ldots,x_n)$ and $g(x_1,x_2,\ \ldots,\ x_n) = 0$ respectively. Then the corresponding Lagrangian function is: $F(x_1,x_2,\ldots x_n,\lambda) = f(x_1,x_2,\ldots,x_n) + \lambda\ g(x_1,x_2,\ldots,\ x_n)$.

If the function F were formed by writing $f-\lambda g$ rather than $f+\lambda g$, the only difference would be a change in the sign of λ (Henderson and Quant; 1980: 381).

First order conditions require the first partial derivatives of F be vanished for both maxima and minima. This condition gives as many equations as there are variables. Setting the above mentioned Lagragian function equal to zero:

$$\frac{\partial F}{\partial x_1} = f_1 + \lambda g_1 = 0$$

$$\frac{\partial F}{\partial x_2} = f_2 + \lambda g_2 = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial x_n} = f_n + \lambda g_n = 0$$

$$\frac{\partial F}{\partial \lambda} = g(x_1, x_2, \ldots, x_n) = 0$$

As there are n choice variables the above system contains n+1 equations and n+1 unknowns. The last equation of the above system ensures the constraint.

Second order conditions require that the quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}\, dx_i\, dx_j$ be negative for maximum and positive for minimum. For a one constraint case they are satisfied if the bordered Hessian determinants $|\bar{H}_2|$, $|\bar{H}_3|$, $|\bar{H}_4|$, $|\bar{H}_5|$, etc. alternate in sign starting from positive. Second order conditions for minimum will be satisfied if they all are negative. The bordered Hessian determinants, i.e. $|\bar{H}_2|$, $|\bar{H}_3|$, $|\bar{H}_4|$, etc., are obtained by bordering the principal minors of the Hessian determinant by a row and a column containing the first partial derivatives of the constraint. The element in the southeast corner of each of these arrays is zero. It is to be noted that the subscript shows the order of Hessian determinant. For instance:

$$|\bar{H}_2| = \begin{vmatrix} F_{11} & F_{12} & g_1 \\ F_{21} & F_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} \; ; \; |\bar{H}_3| = \begin{vmatrix} F_{11} & F_{12} & F_{13} & g_1 \\ F_{21} & F_{22} & F_{23} & g_2 \\ F_{31} & F_{32} & F_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} \quad \text{etc.}$$

Two or More Constraints Case

Let the objective function and the two constraints be $f(x_1, x_2, \ldots, x_n)$, $g^1(x_1, x_2, \ldots, x_n) = 0$ and $g^2(x_1, x_2, \ldots, x_n) = 0$ respectively. The corresponding Lagrangian functions is:

$$F = f(x_1, x_2, \ldots, x_n) + \lambda_1 g^1(x_1, x_2, \ldots, x_n) + \lambda_2 g^2(x_1, x_2, \ldots, x_n)$$

The first order conditions are as usual, that is, to set the first partial derivatives equal to zero. It is shown below:

$$\frac{\partial F}{\partial x_1} = f_1 + \lambda_1 g_1^1 + \lambda_2 g_1^2 = 0$$

$$\frac{\partial F}{\partial x_2} = f_2 + \lambda_1 g_2^1 + \lambda_2 g_2^2 = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial x_n} = f_n + \lambda_1 g_n^1 + \lambda 2 g_n^2 = 0$$

$$\frac{\partial F}{\partial \lambda_1} = g^1(x_1, x_2, \ldots, x_n) = 0$$

$$\frac{\partial F}{\partial \lambda_2} = g^2(x_1, x_2, \ldots, x_n) = 0$$

The last two equations ensure the constraints. The second order conditions are satisfied if the bordered Hessian determinants

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} & g_1^1 & g_1^2 \\ F_{21} & F_{22} & F_{23} & g_2^1 & g_2^2 \\ F_{31} & F_{32}' & F_{33} & g_3^1 & g_3^2 \\ g_1^1 & g_2^1 & g_3^1 & 0 & 0 \\ g_1^2 & g_2^2 & g_3^2 & 0 & 0 \end{vmatrix} ; \begin{vmatrix} F_{11} & F_{12} & F_{13} & F_{14} & g_1^1 & g_1^2 \\ F_{21} & F_{22} & F_{23} & F_{24} & g_2^1 & g_2^2 \\ F_{31} & F_{32} & F_{33} & F_{34} & g_3^1 & g_3^2 \\ F_{41} & F_{42} & F_{43} & F_{44} & g_4^1 & g_4^2 \\ g_1^1 & g_2^1 & g_3^1 & g_4^1 & 0 & 0 \\ g_1^2 & g_2^2 & g_3^2 & g_4^2 & 0 & 0 \end{vmatrix}$$

etc. alternate in sign starting from negative, and those for minimum are satisfied if they are all positive. "If there are m<n constraints, border the principal minors of order m+1 through n with the partial derivatives of the m constraints. The second order conditions for a maximum will be satisfied if the determinants alternate in sign starting with the sign of $(-1)^{m+1}$, and those for a minimum will be satisfied if

all the specified determinants have a sign of $(-1)^m$." (Henderson and Quandt; 1980: 383). The principal minor of order m+1 means that determinant which involves the first m+1 principal diagonal elements of the Hessian determinant.

The classical optimizing technique has some limitations. One weakness is that, "While the first and second-order conditions in terms of derivatives or differentials can normally locate relative or local extreme without difficulty, additional information or further investigation is often required for identification of absolute or global extreme." (Chang: 1984, 431). Over and above this, we will have difficulty in the case of the functions which are not twice differentiable. "The more serious limitation of the calculus approach is its inability to cope with constraints in the inequality form" (Chang; 1984: 431).

A NOTE ON LANGRANGIAN MULTIPLIER

The Lagrangian multiplier has a very important implication. It shows the effect of one unit change in the constant of a constraint on the optimum position. It is remarkable that while interpreting the Lagrangian multiplier it should be taken care of how the Lagrangian function has been formulated. For example, given the objective function $U = f(q_1, q_2)$ subject to the constraint $M = p_1q_1 + p_2q_2$, we can formulate Lagrangian function in four different ways:

1. $L = f(q_1, q_2) + \lambda(M - p_1q_1 - p_2q_2)$
2. $L = f(q_1, q_2) - \lambda(M - p_1q_1 - p_2q_2)$
3. $L = f(q_1, q_2) + \lambda(p_1q_1 + p_2q_2 - M)$
4. $L = f(q_1, q_2) - \lambda(p_1q_1 + p_2q_2 - M)$

In case of 1 and 4, if λ is positive then one unit increase in M leads to an increase in U by the value of λ and vice versa; and, in case of 2 and 3, if λ is positive then one unit increase in M leads to a decrease in U by the value of λ and vice versa. Thus, in the problem of consumers' utility maximization subject to the budget constraint, λ may be interpreted as the marginal utility of money.

CASE OF INEQUALITY CONSTRAINTS

Inequality constraints are more practical than the equality constraints. In consumers' behaviour, if it is assumed that the consumer must spend all his budget M on two commodities Q_1 and Q_2 then the constraint becomes : $M = p_1q_1 + p_2q_2$. However, the consumer may be expected to spend less than M, in that case the constraint becomes: $p_1q_1 + p_2q_2 \leq M$. Similarly, if restriction is imposed upon the consumer to spend at least M amount of budget then the constraint becomes: $p_1q_1 + p_2q_2 \geq M$. In production also inequality constraints may arise in different ways. If a firm is asked to produce at least 200 combined units of X_1 and X_2 then the constraint becomes $X_1 + X_2 \geq 200$. Similarly, if a firm has Rs. 1000 to invest on two products Q_1 and Q_2, then the constraint becomes: $c_1q_1 + c_2q_2 \leq 1000$, where c_i is per unit capital cost on one unit production of Q_i. Thus, if the assumption of equality constraint is relaxed, then three types of constraints are possible, namely, in equality form, in less than or equal to form and in greater than or equal to form. We may think of the possibility of the constraints in strict inequalities.

Optimization with Inequality Constraint

In the case of inequality constraints the same first-order and second-order conditions are to be satisfied. The only difference lies in the way of tackling the constraints. Suppose that we have an objective function and constraints as follows:

$$\text{optimize} \quad F=f(x_1, x_2, \ldots\ldots, x_n)$$

$$\begin{aligned} \text{subject to} \quad & g^1(x_1, x_2, \ldots\ldots, x_n) \leq M_1 \\ & g^2(x_1, x_2, \ldots\ldots, x_n) \geq M_2 \\ & g^3(x_1, x_2, \ldots\ldots, x_n) \leq M_3 \\ & \vdots \\ & g^m(x_1, x_2, \ldots\ldots, x_n) \geq M_m \\ & \quad x_i \geq 0 , \quad i = 1,2,\ldots\ldots,n \end{aligned}$$

Here we have n choice variables and m constraints. In the case of equality constraints it is required that n should be greater than m. But in the case of inequality constraints n may be less than m, greater than m, or may be equal to m.

The Solution

The solution to the above problem is not straight forward as compared to the problem with equality constraints. In the above problem constraints may be inconsistent; finite optimization may not be possible even if the constraints are consistent; there may be interior solution; or, there may be boundary point solution. Concerning the consistency of the constraints, given the set of whole constraints if any two of them are inconsistent then the whole should be treated as inconsistent. Sometimes we can easily detect inconsistency by observing the constraints. But this is possible only in the case of small number of choice variables.

To solve the above problem it is essential first to check whether the constraints impose a real limitation or not. Thus our first step is to try to optimize without constraints. It means, our problem is:

$$\text{Optimize (max or min): } F=f(x_1, x_2, \ldots\ldots, x_n),\ x_i \geq 0.$$

This may or may not give free optimum, i.e. optimum without constraint, which depends upon the convexity or concavity of the objective function. A strictly concave objective function gives free maxima and a strictly convex objective function gives free minima. In this connection there are different possibilities which we shall explain pointwisely.

1. If the objective function gives free optimum point then we check whether the constraints are satisfied at that point or not. If it is found that all the constraints are satisfied at that point then we conclude that the constraints impose no real limitation and the optimum point without any constraint is the required solution.

Example 1. Max: $z = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5$

Subject to: $x + y \leq 10$

$3x + 2y \leq 20$

$xy \geq 12$

$x, y \geq 0$

If we try to maximize this objective function without constraint, then we get maxima at ($\bar{x} = 4$, $\bar{y} = 3$) where all the constraints are satisfied. Thus we conclude that the constraints impose no real limitation and the point ($\bar{x} = 4$, $\bar{y} = 3$) is the solution.

2. Sometimes it is possible that free optimum exists but constraints are not satisfied. It means that the constraints impose a real limitation and there is also a possibility of inconsistency of constraints. Practically, the chances of inconsistency increases as the number of constraints increases. To solve the problem in this situation, we first distinguish those constraints which are not satisfied at the point of free optimum. Then we form different Lagrangian functions taking only one constraint, in equality, in each; and, we try to solve them. This gives us as many solutions as the number of limiting constraints. Then at every solution we have to check whether all the constraints are sarisfied or not. If all the constraints are found to be satisfied at only one point, obviously that very point is the final solution. In case it is found that all the constraints are satisfied at more than one point then the best desirable is selected by comparing values at different points.

If none of the solutions given by the Lagrangian functions with one constraint satisfies all the constraints, then we should proceed to form all the possible Lagrangian functions with two independent and consistent constraints, in equalities, in each. Again, we try to solve those Lagrangian functions and try to find final solution. If it, again, does not give solution, we proceed to form all the possible Lagrangian functicns with three independent and consistent constraints; and try to solve them. This process should be continued until we find that solution, where all the constraints are satisfied; or, until the number of the constraints in each Lagrangian function reaches n-1, where n is the number of the choice variables. The reason behind doing this is that given a set of simultaneous equations there cannot be more than that number of independent equations as there are choice variables; therefore, we have at least one equation from objective function. If it still does not give solution satisfying all the constraints, we try for corner point solution. If it is still impossible to find solution then we conclude that constraints are not consistent.

Example 2. Max: $z = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5$

subject to: $x + y \leq 6$

$2x + y \leq 10$

$x + 3y \leq 14$

$xy \geq 5$

$x, y \geq 0$

Here, the objective function gives free maxima at $(\bar{x} = 4, \bar{y} = 3)$, where the first two constraints are not satisfied meaning that they are limiting constraints. Now, our task is to form two separate Lagrangian functions:

$$L = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5 + \lambda\ (6-x-y)$$

$$L = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5 + \lambda\ (10-2x-y)$$

Equation 1st is maximum at $(\bar{x} = 3.8, \bar{y} = 2.2)$ and equation 2nd is maximum at $(\bar{x} = 3.83, \bar{y} = 2.33)$. At the point $(\bar{x} = 3.8, \bar{y} = 2.2)$ all the constraints are satisfied; and at the point $(\bar{x} = 3.83, \bar{y} = 2.33)$ the first constraint is not satisfied. Thus the point $(\bar{x} = 3.8, \bar{y} = 2.2)$ is the final solution.

Example 3. Max: $z = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5$

subject to: $x + y \leq 6$

$2x + y \leq 10$

$xy + \geq 12$

$x, y \geq 0$

Here, free maximum is found at $(\bar{x} = 4, \bar{y} = 3)$, where the first two constraints are not satisfied. Thus we form two separate Lagrangian functions:

$$L = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5 + \lambda\ (6-x-y)$$

$$L = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5 + \lambda\ (10-2x-y)$$

As earlier, equation 1st is maximum at ($\bar{x} = 3.8$, $\bar{y} = 2.2$) where third constraint is not satisfied; equation 2nd is maximum at ($\bar{x} = 3.83$, $\bar{y} = 2.33$) where both first and third constraints are not satisfied. It means that, we should proceed to try for corner point solution. In this particular problem we now need to find all the corner points given by each constraint and intersection of constraints. But, as the constraints first and third are inconsistent the set of whole constraints are also inconsistent.

Example 4. Minimize: $\Pi = 7Q_1^2 - 2Q_1Q_2 + Q_2^3$

$$\text{Subject to: } Q_1 + Q_2 \leq 400$$

$$Q_1Q_2 \geq 200$$

$$Q_1, Q_2 \geq 0$$

Here, the objective function gives free minima at two points ($\bar{Q}_1=0$, $\bar{Q}_2=0$) and ($\bar{Q}_1 = \frac{1}{147}$, $\bar{Q}_2 = \frac{7}{147}$), both of which violates the second constraint. Then, we proceed to form the following Lagrangian function:

$$L = 7Q_1^2 - 2Q_1Q_2 + Q_2^3 + \lambda\ (200 - Q_1Q_2)$$

This function gives minima at ($\bar{Q}_1=20.28$, $\bar{Q}_2=9.86$), where both the constraints are satisfied and $\lambda = 28.8$. Thus, this point is the required solution.

3. Problems become more complicated if the objective functions cannot be optimized freely. However, even if there is no free optimum, constraint optimization may be possible. The reason is that, even if a function is not strictly concave or convex it may be strictly quasi-concave or quasi-convex.

In case the objective function does not have free optima of our interest, then our task is to form different Lagrangian functions with only one constraint, in equality, in each. Then we try to optimize each of them with the help of the first-order conditions and the second-order conditions. Nevertheless, it may or may not be possible to find solution of our interest which depends upon the strictly quasi-concavity or strictly quasi-convexity of the objective function. If the Lagrangian functions with one constraint give solutions, then we check whether all the constraints at different points of the solution are satisfied or not. The technique is, now, same as in point 2.

It is very important that, at each solution we should check the value of the Lagrangian multiplier in order to know the effect of the strictly inequality part of the constraints.

Example 5. Max: $z = xy$

Subject to: $x + y \leq 5$

$2x + 4y \leq 15$

$x + 3y \geq 25$

$x, y \geq 0$

Here, the objective function do not have free maxima. Therefore, it is required to form the following three Lagrangian functions:

$$L = xy + \lambda\ (5-x-y)$$

$$L = xy + \lambda\ (15-2x-4y)$$

$$L = xy + \lambda\ (25-x-3y)$$

Solving these equations shows that equation 1st is maximum at ($\bar{x} = 2.5$, $\bar{y} = 2.5$), which violates third constraint; equation 2nd is maximum at ($\bar{x} = 3.75$, $\bar{y} = 1.875$), which violates both the first and the third constraints; and, equation 3rd is maximum at ($\bar{x} = 12.5$, $\bar{y} = 4.166$), which violates both the first and the second constraints. Since at no point all the constraints are satisfied, we should proceed to find all the possible corner points. In this particular example the constraints first and third are inconsistent. Thus we need not proceed as the constraints are inconsistent.

Example 6. Max: $z = xy$

Subject to: $x + y \leq 5$

$2x + 4y \leq 15$

$x, y \geq 0$

This problem is same as the previous one, only the third constraint has been removed. As there is no free maxima we need to form two Lagrangian functions as follows:

$$L = xy + \lambda\ (5-x-y)$$

$$L = xy + \lambda\ (15-2x-4y)$$

As before, equation 1st is maximum at ($\bar{x} = 2.5$, $\bar{y} = 2.5$) and equation 2nd at ($\bar{x} = 3.75$, $\bar{y} = 1.875$). Clearly, only at point ($\bar{x} = 2.5$, $\bar{y} = 2.5$) both the constraints are satisfied. At this point $\lambda = 2.5$ implying that as the constant of the first constraint decreases by one unit it will reduce the value of objective function at the point of optimum by approximatly 2.5 units.

Thus, the point ($\bar{x} = 2.5$, $\bar{y} = 2.5$) is the required solution.

4. If the objective function cannot be optimized freely and also with constraints, then we seek corner point solution. Such cases usually come when we try to maximize strictly convex or strictly quasi-convex function; or, if we try to minimize strictly concave or strictly quasi-concave function. In such cases we first find corner points given by each constraint and intersection of constraints; and we try to find that point where all the constraints are satisfied and gives best desirable value of the objective function.

Sometimes, especially when objective function and constraints are in complicated form, it is desirable to check corner points as the solution. Although it is not sure if it gives the solution, yet it may, sometimes, save time and resources significantly.

Example 7. Max: $\Pi = x_1^2 + x_2^2 - 4x_2 + 4$

subject to: $5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

The objective function gives free minima at ($\bar{x}_1 = 0$, $\bar{x}_2 = 2$), which is not of our interest. As the function is strictly concave, it is also strictly quasi-concave. Thus, for maximization we seek corner point solution. Two corner points given by the constraint are ($\bar{x}_1 = 0$, $\bar{x}_2 = 5$) and ($\bar{x}_1 = 3$, $\bar{x}_2 = 0$). Value of Π at ($\bar{x}_1 = 0$, $\bar{x}_2 = 5$) is 9 and that at ($\bar{x}_1 = 3$, $\bar{x}_2 = 0$) is 13. Thus the point ($\bar{x}_1 = 3$, $\bar{x}_2 = 0$) is the solution.

Example 8. Min: $c = x_1 + x_2$

Subject to: $x_1^2 + x_2 \geq 9$

$x_1 x_2 \leq 8$

$x_1, x_2 \geq 0$

Here, the objective function is linear. We may say that c is minimum at ($\bar{x}_1 = 0$, $\bar{x}_2 = 0$), but the first constraint is not satisfied. Then we form two Lagrangian functions as follows:

$$L = x_1 + x_2 + \lambda\ (9 - x_1^2 - x_2)$$

$$L = x_1 + x_2 + \lambda\ (8 - x_1 x_2)$$

Equation 1st gives maxima at ($\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{35}{4}$), which is not of our interest. Equation 2nd gives minima at ($\bar{x}_1 = \sqrt{8}$, $\bar{x}_2 = \sqrt{8}$), where $\lambda = \frac{1}{\sqrt{8}}$ showing that one unit reduction in the value of the constraint will lead to a reduction of the value of c. Thus, $(\bar{x}_1 = \sqrt{8}, \bar{x}_2 = \sqrt{8})$ is not the required solution. Here also we should seek corner point solution. The corner points given by the first constraint are $(\bar{x}_1 = 0, \bar{x}_2 = 9)$ and $(\bar{x}_1 = 3, \bar{x}_2 = 0)$; and the corner points given by the intersection of the constraints are $(\bar{x}_1 = 1, \bar{x}_2 = 8)$ and $(\bar{x}_1 = 2.37, \bar{x}_2 = 3.38)$. At every corner point all the constraints are satisfied. The comparison shows that c is minimum at $(\bar{x}_1 = 3, \bar{x}_2 = 0)$, which is the required solution.

Example 9. Minimize: $z = 6\, Q_1 Q_2^{\,2}$

$$\text{Subject to: } 2\, Q_1^2 + Q_2 \geq 50$$

$$Q_1 \leq 10$$

$$Q_1, Q_2 \geq 0$$

Here, if we directly try corner point solution we find that z = 0 at $(\bar{Q}_1 = 5, \bar{Q}_2 = 0)$, where all the constraints are satisfied. Not only this, when $\bar{Q}_2 = 0$ any value of Q_1 in the interval $(5 \leq Q_1 \leq 10)$ gives minima. If we minimize the objective function freely, first constraint is not satisfied at $(\bar{Q}_1 = 0, \bar{Q}_2 = 0)$. The Lagrangian function with the first constraint maximizes z at $(\bar{x}_1 = \sqrt{5}, \bar{x}_2 = 40)$, which is not of our interest.

Example 10. Max: $\Pi = Q_1^3 + Q_1 Q_2$

$$\text{Subject to: } Q_1 + Q_2^5 \leq 10$$

$$Q_1^2 + 2Q_2 \geq 4$$

$$Q_1, Q_2 \geq 0$$

Here, the objective function does not give free maxima; and corresponding Lagrangian functions take complicated forms. Thus, it is desirable to try for corner point solution. If we check all the corner points. We find that, at point $(\bar{Q}_1 = 10, \bar{Q}_2 = 0)$ Π is maximum and both the constraints are also satisfied.

Example 11. Max: $\Pi = 7Q_1^2 - 2Q_1Q_2 + Q_2^3$

subject to: $Q_1 + q_2 \leq 400$

$Q_1Q_2 \geq 200$

$Q_1, Q_2 \geq 0$

Here, the objective function does not give free maxima. Also, the Lagrangian function with the first constraint gives two extreme points, both being minimum points. Let us now check the corner points for possible solution. The first constraint gives two corner points ($\bar{Q}_1 = 0$, $Q_2 = 400$) and ($\bar{Q}_1 = 400$, $\bar{Q}_2 = 0$), both of which violate the second constraint. The intersection of two constraints gives two corner points ($\bar{Q}_1 = 0.50063$, $\bar{Q}_2 = 399.49937$) and ($\bar{Q}_1 = 399.49937$, $\bar{Q}_2 = 0.50063$). Checking the values of Π at these points shows that Π is maximum at ($\bar{Q}_1 = 0.50063$, $\bar{Q}_2 = 399.49937$). The same problem was minimum at ($\bar{Q}_1 = 20.28$, $\bar{Q}_2 = 9.86$) as shown in Eg.-4.

Example 12. Max: $z = 60x + 34y - 4xy - 6x^2 + 5$

: $y + 7x \leq 23$

$y - \frac{1}{2}x \leq 1$

$x, y \geq 0$

Here, the free maxima violates the first constraint. Corresponding Lagrangian functions also do not give acceptable solutions. Thus, we need to try corner solution. The corner points where all the constraints are satisfied are ($\bar{x} = \frac{23}{7}$, $\bar{y} = 0$), ($\bar{x} = 0$, $\bar{y} = 1$) and ($\bar{x} = 3$, $\bar{y} = 2$). A comparison shows that z is maximum at ($\bar{x} = 3$, $\bar{y} = 2$).

CONCLUSION

Many problems concerning optimization with inequality constraints may be tackled by modifying the classical technique of optimization. But actually the solution of the optimization problems with inequality constraints is very complicated task. The techniques worked out here are very useful in solving many problems. Sometimes, because of the complicated forms of the equations, i.e. objective function and constraints, solutions will be very difficult. Although we have techniques, we may not be in the position to obtain final solution. Sometimes, there may be more than one solution or infinite number of solutions which we may fail to detect. There should be further investigations on how the much complicated problems may be tackled in the simplest possible way.

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