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Some Theoretical Considerations of Optimal Taxation in the Light of Economic Efficiency Criterion

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INTRODUCTION

Optimal tax theory is one of the most controversial topics in the theory of public finance. Ever since the publication of Adam Smith's The Wealth of Nations in 1776, economists are arguing on the different aspects of taxation. It is not surprising to notice that the title of David Ricardo's famous book is The Principles of Political Economy and Taxation (1817). Taxation directly affects the economic well-being of the citizens. It is natural for the economists as well as ordinary people to have interest in finding an optimum tax structure in the economy.

One of the hotly debated issues is direct versus indirect taxation. Alfred Marshall developed a concept of consumer surplus to analyse the economic effects of different taxes. He demonstrated that a commodity tax reduces consumer surplus by more than the gross tax yield. He wrote, "For on that part of the consumption of the commodity which is maintained, the consumer loses what the State receives; and on that part of the consumption which is destroyed by the rise in price, the consumers' surplus is destroyed; and of course there is no payment for it to the producer or to the State."[1]

J.R. Hicks, by defining consumer surplus as the "compensating variation in income, whose loss would just offset the fall in price, and leave the consumer no better off than before", proved that "a tax on commodities lays a greater burden on consumers than on income tax."[2]

Later, attention was drawn to the fact that the income tax has a similar, if not exact, excess burden compared to a lump-sum tax, as it discriminates in favour of taking leisure. Commenting on the Classical "excess burden" doctrine, H.P. Wald writes, "It fails to recognize that whatever merit the "excess burden" doctrine might have as a guide to tax policy is seriously weakened, by the fact that the doctrine is just as applicable to many direct methods of taxation, such as individual income taxes, as to indirect methods."[3]

It has been observed that high marginal rates of income tax can have adverse effects in reducing output by decreasing the incentives to work. The income tax, however, was still the preferred tax, until I.M.D. Little's article, "Direct versus Indirect Taxes"[4] was published in 1951. Analysing

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a three good model, he demonstrated that there is nothing to choose between direct and indirect taxes in that both give rise to an excess burden on the taxpayers above the receipts to the State, unless further knowledge about the interrelationship of the three goods can be found.

The argument against indirect tax in favour of direct one is unsatisfactory because it assumes that the resources used are identical in both cases. In other words, it has to be assumed, for the validity of the argument, that the supply of labour is the same whether the taxation is direct or indirect. This amounts to saying that, for every consumer, the cross-elasticity of demand for leisure with respect to all other prices is zero, i.e., leisure is not substitutable for any other good. If the supply of labour is allowed to vary, the argument against indirect taxation is not perfectly general. Corlette and Hague[5] proved that, under certain conditions, a change from an income tax to a system of indirect taxes can really increase the supply of labour and raise real income, the tax revenue remaining the same.

The recent literature on optimal taxation can be taken as an attempt to clarify the structure of the arguments advanced to support changes in the tax system, tracing the implications of taxes and quantifying the trade-offs between the various objectives of tax policy. It has examined the optimal structure for certain types of taxation taken in isolation, such as the optimal income taxation and the optimal commodity taxation, without arguing which type is better than the other.

This paper is confined to the discussion of optimal commodity taxation in the light of economic efficiency criterion, without considering its distributional characteristic. It contains five sections. The first section introduces the tax rules. The second section presents a basic model of optimal commodity taxation and reformulates the tax rules on the basis of general equilibrium analysis. The third section discusses the uniformity issue and presents some special cases of the basic model. The fourth section examines the inverse elasticity rule on the basis of general equilibrium analysis and discusses it in the context of three-goods economy. The fifth section makes an improvement on the basic model taking into consideration of production conditions and production efficiency, and gives a proof of the statement that the production can affect the optimum tax structure under general conditions.

TAX RULES

The problem of finding an optimum commodity tax structure has been discussed intensively for more than half a century in the theory of public finance and taxation. One of the first contributions appeared in Marshall's Principles of Economics, first published in 1890. The most recent contributions can be found in the economic journals published since 1970.

Marshall found that the net loss of consumer surplus is minimum, "for those commodities the demand for which is most inelastic, that is necessaries."[5] According to his analysis based on the cardinal utility and constant marginal utility of money assumptions, the commodity which is to be taxed at the highest rate is the one with the lowest price-elasticity of demand.

Based on Marshallian partial equilibrium model of a single market, R.L. Bishop[6] measured the total deadweight loss for a given revenue, T, as follows:

Assume that: (1) except for one taxed industry, the prices of all goods are equal to their marginal costs and the prices of all factors are equal to the values of their marginal products; (2) the taxed industry is a small portion of the whole economy so that changes in its output and price have only negligible effects on goods and factor prices in the rest of the economy; and (3) the demand and supply curves for the taxed good reflect only negligible income effects.

In the absence of any tax, a perfectly competitive industry with the demand and supply schedules, represented by D and S respectively in the Figure I, is in equilibrium at the point $P(x_o, p_o)$. If a specific tax is now imposed, in the amount of $t = P_1 - p_1$, the supply curve shifts up from SS to S'S'. As a result, output is reduced to x_1, demand price is raised to P_1, and supply price is lowered to p_1. The total cost incurred to the consumers is then measured by the trapezoid lying between the price axis and the demand schedule, over the price range from p_o to P_1; and the total cost incurred to the producers is similarly measured by the trapezoid lying between the price axis and the supply schedule over the price range from p_1 to p_o. Since the total tax revenue corresponds to the rectangular area, $(P_1 - p_1)x_1$, this leaves the dead-weight loss as the shaded triangular area between the demand and supply schedules over the output range from x_1 to x_o.

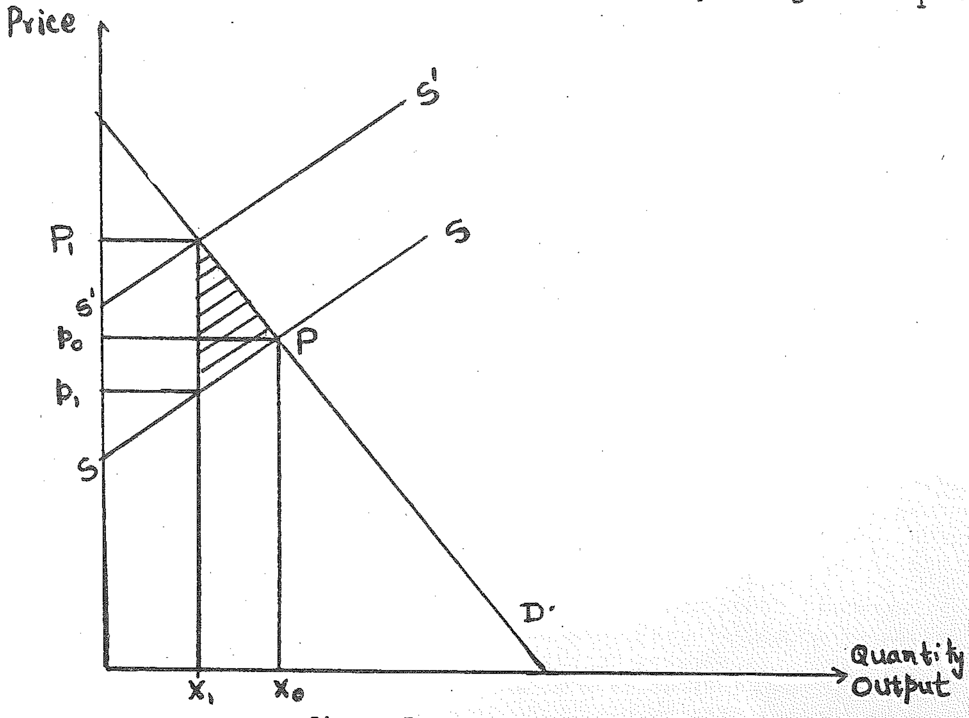

Figure I

In this simple case where the demand and supply schedules are both linear, the ratio of the dead-weight loss (L) to the tax revenue (T) is independent of the demand and supply slopes, but depends only on the ratio that x_1 bears to x_o. Thus, since

$$L = \frac{(P_1 - p_1)\,(x_o - x_1)}{2}\ , \quad \text{and } T = (P_1 - p_1)x_1\ ,$$

it may be established that

$$\frac{L}{T} = \frac{(x_o - x_1)}{2x_1} = \frac{1 - (x_1/x_o)}{2(x_1/x_o)}$$

$$\frac{dL}{dT} = \frac{(x_o - x_1)}{2x_1 - x_o} = \frac{1 - (x_1/x_o)}{2(x_1/x_o) - 1}\ .$$

The dependence of L/T and dL/dT on just x_1/x_o has the interesting result that, if there are several competitive industries with linear demand and supply, and if a given amount of tax is to be collected by taxing their outputs, the total dead-weight loss will be minimized if the tax rates are such as to reduce those outputs by the same percentage. This is so because dL/dT is the same in each taxed industry when x_1/x_o is the same; and, since dL/dT is a decreasing function of x_1/x_o, that is the condition for minimizing aggregate L for given aggregate T.

Ursula K. Hicks[7] has also produced similar formula to measure dead-weight loss, but it is expressed in terms of the elasticities of demand and supply. For an indefinitely small tax, her measure of the total dead-weight loss for a given revenue (T) is:

$$\frac{T^2}{2Px\ (1/E_d + 1/E_s)}$$

where E_d and E_s denote the elasticities of demand and supply, and Px denotes expenditure on the commodity, P and x representing its price and quantity respectively.

Thus to minimize price-distortion we should tax those goods which (i) have a low price elasticity of demand, (ii) have a low price elasticity of supply, and (iii) form an important part of consumers' budgets.

In his pioneering work, "A Contribution to the Theory of Taxation",[8] published in 1927, F.P. Ramsey produced a tax rule which is commonly known as Ramsey rule. For an optimality, he proved that in raising a revenue by proportionate taxes on given commodities the taxes should be such as to reduce in the same proportion the production of each commodity taxed.

Atkinson and Stiglitz interpreted the Ramsey rule in this way: "In the Ramsey case, we wish to minimize the total dead weight loss over all taxable commodities, so that for each commodity the marginal dead weight loss associated with raising a marginal dollar of tax revenue must be the same. In the case of a perfectly elastic supply this requires (for small taxes)

$$\frac{t_i}{q_i} D_d^i = \text{constant for all commodities } i = 1, \ldots, n,$$

or that the (ad valorem) tax rates be inversely proportional to the elasticity of demand in each industry."[9]

Nevertheless, the partial equilibrium analysis of the commodity taxation is not quite satisfactory in view of the restrictive assumptions upon which it is based. In particular, it requires (i) the absence of income effects, and (ii) the independence of demand functions. There has, therefore, been considerable scepticism about its applicability in practical tax policy. A.R. Prest for example, rejects these tax rules on the ground that, "such restrictive assumptions have to be made in order to derive a solution that they appear to have little practical significance", although he offers nothing to replace them. However, in 1951, Samuelson submitted to the U.S. Treasury a paper containing a generalization of Ramsey rule "namely, that the optimal pattern is the one at which the response of all goods and factors to a further compensated-Slutsky price distortion would in equal percentage (virtual) reductions."[11]

Since 1970 there has been a general revival of interest among the economists in the subject of optimal taxation. A series of articles have been published dealing with optimal tax theory under a variety of assumptions. Among them the article[12] of P.A. Diamond and J.A. Mirrlees particularly represents a major generalization and extension of Ramsey rule.

In 1976, Agnar Sandmo made an introductory survey of the field and commented, "The field now seems well established as one of considerable interest both from a theoretical and a practical point of view, and...".[13] He also produced a basic model of optimal commodity taxation, which is presented below.

BASIC MODEL OF OPTIMUM COMMODITY TAXATION

Suppose there are m+1 commodities in an economy, the first of which is labour denoted by good 0 and the remaining m commodities are consumer goods. The latter are subject to commodity taxation.

Suppose that the government has a fixed revenue constraint. The government wishes to collect that amount of revenue, T, by taxing consumer goods. If t_i is the tax per unit of commodity i, x_i being its quantity, the fixed revenue constraint can be expressed as,

$$\sum_{i=1}^{m} t_i \cdot x_i = T \qquad \ldots \ldots (1)$$

Let p_i and P_i be the producer and consumer prices, respectively, of good i. The producer prices are assumed to be fixed so that the problem of selecting a tax structure is equivalent to choosing a structure of consumer prices. Thus,

$$P_i = p_i + t_i \qquad \ldots \quad \ldots \ (2)$$

It is assumed that good 0, (i.e., labour) is not taxed and is used as numériare, so that $P_o = p_o = 1$ and $t_o = 0$. ... ... (3)

The general assumption is that the government maximizes a utility function which is individualistic and impersonal like one belonging to the Bergson - Samuelson family:

$$W = W\ (U_1, \ldots, U_n) \qquad \ldots \quad \ldots \ (4)$$

where U_j is the utility function of individual j, expressed as

$$Uj = Uj\ (xj_o, \ldots, x_{jm}) \qquad \ldots \quad \ldots \ (5)$$

In order to focus our attention on the economic efficiency aspect only, without considering distributional aspects, we simply assume that all consumers are identical and can be represented by a single consumer.

Now this representative consumer maximizes his utility function,

$$U = U\ (x_o, x_1, \ldots, x_m) \qquad \ldots \quad \ldots \ (6)$$

subject to his budget constraint,

$$\sum_{i=o}^{m} P_i x_i = I = 0, \qquad \ldots \quad \ldots \ (7)$$

(assuming no lump-sum transfer payments to the consumer)

Notice, here, that this consumer budget constraint has already taken into account the income the consumer earns from his supply of labour. It can be seen easily if we think of labour supply as being measured negatively.

This yields the first order conditions for the consumer equilibrium:

$$\frac{\partial U}{\partial x_i} = \lambda\ P_i, \quad i=o,i, \ldots m, \qquad \ldots \quad \ldots \ (8)$$

and the demand functions: $x_i = x_i(P_o, P_1, \ldots P_m, I)$ ... ... (9)

Here, λ is the Lagrange multiplier and can be interpreted as the marginal utility of income.

In order to minimize the dead-weight loss to the consumer due to taxation, the government must choose a tax structure $(t_1, t_2, \ldots t_m)$ subject to its revenue constraint (1) in such a way as to maximize social welfare function, now represented by (6). This problem can be formulated in terms of a Lagrange function,

$$L = U(x_o, x_1, \ldots, x_m) + \mu\left(\sum_{i=1}^{m} t_i x_i - T\right) \qquad \ldots \quad \ldots (10)$$

Setting the partial derivatives of the Lagrange function (10) with respect to $t_k = P_k - p_k$ equal to zéro, we get the necessary conditions for a constrained maximization of social welfare:

$$\sum_{i=o}^{m} \frac{\partial U}{\partial x_i} \cdot \frac{\partial x_i}{\partial P_k} + \mu\left(\sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial P_k} + x_k\right) = 0,\ k = 1,2,\ldots m \qquad \ldots \quad \ldots (11)$$

Substituting the values of $\frac{\partial U}{\partial x_i}$ from (8) into (11), we get

$$\lambda \sum_{i=o}^{m} P_i \frac{\partial x_i}{\partial P_k} + \mu\left(\sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial P_k} + x_k = 0,\ k = 1,2,\ldots,m \qquad \ldots \quad \ldots (12)$$

Differentiating both sides of equation (7) with respect to P_k, we get

$$\sum_{i=o}^{m} P_i \frac{\partial x_i}{\partial P_k} + x_k = 0,\ k = 1,2,\ldots,m \qquad \ldots \quad \ldots (13)$$

Utilizing (13), we can rewrite the conditions as

$$-\lambda x_k + \mu\left(\sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial P_k} + x_k\right) = 0$$

and finally as

$$\sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial P_k} = -\left(\frac{\mu-\lambda}{\mu}\right) x_k = -\nu x_k,\ k = 1,2,\ldots m \qquad \ldots \quad \ldots (14)$$

where $\nu = 1 - \frac{\lambda}{\mu}$.

From the theory of consumer demand, we can derive Slutsky equation by taking the derivatives of the demand functions (9) with respect to consumer price P_k,

$$\frac{\partial x_i}{\partial P_k} = -x_k \frac{\partial x_i}{\partial I} + S_{ik} \quad i, k = 1,2,\ldots, m \qquad \ldots \quad \ldots (15)$$

where I is consumer's income and S_{ik} is the substitution effect.

Utilizing Slutsky equation (15), we can rewrite the condition (14) as

$$\sum_{i=1}^{m} t_i S_{ik} = - \nu x_k + x_k \sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial I} \quad \ldots \ldots (16)$$

But the substitution effects are symmetric, $S_{ik} = S_{ki}$ ... ... (17)
so the conditions can be expressed as

$$\frac{\sum_{i=1}^{m} t_i S_{ki}}{x_k} = - \nu + \sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial I}, \quad k = 1, \ldots, m. \quad \ldots \ldots (18)$$

Since the right hand side of this expression (18) is independent of k, the condition for the optimality of commodity taxes requires that the percentage reductions in compensated demand for all commodities should be equal.

This is a Ramsey sort of result obtained on the basis of general equilibrium analysis. This result is particularly valuable when contrasted to the erroneous view that uniform commodity taxation is best from efficiency aspects. However, it can hardly be applied in actual tax policy recommendations. As it stands, it is valid only for an infinitesimal tax revenue. But there could have an interesting version of this Ramsey result if it were true that

$$\frac{\partial x_i}{\partial P_k} = \frac{\partial x_k}{\partial P_i}, \quad i, k = 1, \ldots, m, \quad \ldots \ldots (19)$$

for it could then possible to express conditions (14) as

$$\left(\sum_{i=1}^{m} t_i \frac{\partial x_k}{\partial P_i} \right) / x_k = - \nu, \quad k = 1, \ldots, m \quad \ldots \ldots (20)$$

which implies that the Ramsey result of equal percentage reduction in compensated demand would hold without the restrictive assumption of zero tax revenue.

Now, by the symmetry of the substitution effects (17), we have

$$\frac{\partial x_i}{\partial P_k} + x_k \frac{\partial x_i}{\partial I} = \frac{\partial x_k}{\partial P_i} + x_i \frac{\partial x_k}{\partial I} \quad \ldots \ldots (21)$$

If equation (19) is true, then $\frac{1}{x_i} \frac{\partial x_i}{\partial I} = \frac{1}{x_k} \frac{\partial x_k}{\partial I}$

Finally, multiplying both sides by income I, we find that equation (19) implies

$$\frac{I}{x_i}\frac{\partial x_i}{\partial I} = \frac{I}{x_k}\frac{\partial x_k}{\partial I}, \quad i, k = 1, \dots, m \qquad \dots \dots (22)$$

i.e., equal income elasticities for all taxed goods.

This interesting result can be illustrated diagrammatically for the simple case of two taxed goods. The condition (22) requires the indifference map to be homothetic so that there should be equal proportionate reduction in demand for the taxed goods along the line 0Q, as depicted in the Figure II below. It implies uniform taxation, i.e., no change of relative prices within the group of taxed goods.

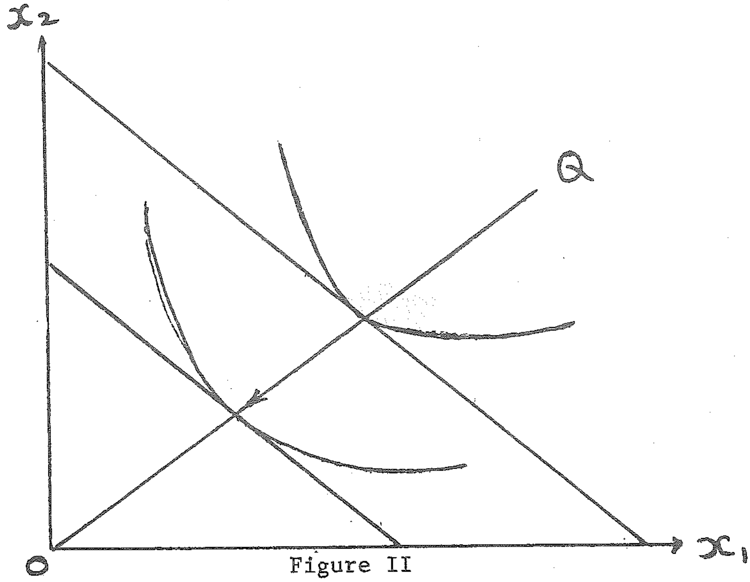

Figure II

The above hypothesis suggests that the deviations from the rule of uniform proportional reduction in demand for all taxed goods must be sought in their unequal income elasticities. Another interesting result to this effect can in fact be found as follows:

Rearranging the terms in equation (21) as

$$\frac{\partial x_i}{\partial P_k} = \frac{\partial x_k}{\partial P_i} + x_i\frac{\partial x_k}{\partial I} - x_k\frac{\partial x_i}{\partial I} \qquad \dots \dots (23)$$

and substituting the value of $\frac{\partial x_i}{\partial P_k}$ obtained from (23) into (14), we get

$$\sum_{i=1}^{m} t_i \left(\frac{\partial x_k}{\partial P_i} + x_i\frac{\partial x_k}{\partial I} - x_k\frac{\partial x_i}{\partial I}\right) = -\nu x_k, \; k=1, \dots, m$$

or $$\left(\sum_{i=1}^{m} t_i \frac{\partial x_k}{\partial P_i}\right) / x_k = -\nu - \sum_{i=1}^{m} t_i x_i \left(\frac{1}{x_k}\frac{\partial x_k}{\partial I} - \frac{1}{x_i}\frac{\partial x_i}{\partial I}\right) \qquad \dots \dots (24)$$

Thus, if the proportionate change in demand for commodity k resulting from a hypothetical change in exogeneous income is higher on the average, using tax payments as weights, than for the other taxed commodities, then this implies a larger than average proportionate reduction of demand.

This result is now applicable as a tax policy recommendation. Tax changes have both income and substitution effects, and the income effects are just like the changes that would have brought if the revenue had been collected by lump-sum taxes. Since the latter effects are non-distortionary, so are the pure income effects. Therefore tax policy should reduce the demand most for the commodities in which income effects dominate the substitution effects.

THE UNIFORMITY ISSUE

One would probably argue that if only the set of taxed goods were extended so as to include labour in the basic model of optimal commodity taxation, then uniform taxation would lead to optimality, since this would imply that no relative prices in the system would be changed, as compared with the pre-tax Pareto optimal equilibrium. One such argument is produced by Avinash Dixit in his model as follows:[14]

If all the commodities including labour are taxable, we get by differentiating the consumer budget constraint (7) with respect to P_k,

$$\sum_{i=o}^{m} P_i \frac{\partial x_i}{\partial P_k} + x_k = 0 , \quad k = o, 1, \ldots m. \qquad \ldots \ldots (25)$$

Substituting the value of x_k from (25) into (14), we get

$$\sum_{i=o}^{m} (t_i - \nu P_i) \frac{\partial x_i}{\partial P_k} = 0 , \quad k = o, 1, \ldots m. \qquad \ldots \ldots (26)$$

In this way, Dixit concludes that a proportional tax structure, i.e., ($t_i = \nu P_i$) is optimum. Moreover, according to his analysis, if the matrix $(\frac{\partial x_i}{\partial P_k})$ is non-singular, it is the only optimum structure.

Criticizing the Dixit result, Sandmo shows that taxing all commodities at the same rate is meaningless in a general equilibrium model where Walras's law holds, for the simple reason that such a tax structure would result in zero tax revenue. For if $t_i/P_i = \nu$, a constant, then

$$T = \sum_{i=o}^{m} t_i x_i = \nu \sum_{i=o}^{m} P_i x_i \qquad \ldots \ldots (27)$$

But, from the consumer budget constraint (7), T is necessarily zero. Taxing all commodities while retaining all relative prices constant means subsidizing labour supply at the same rate, and taxes and subsidies must necessarily cancel each other, and the government's target to raise a fixed revenue T is not fulfilled.

The possible optimality of uniform taxation cannot be established in this way. However, it is certainly of great interest to look for special cases in which the uniform taxation is optimal.

Assume again that commodity 0 (i.e., labour) is not taxed and is used as numéraire, so that $P_o = p_o = 1$ and to $= 0$.

We can rewrite equation (13) as

$$\frac{\partial x_o}{\partial P_k} + \sum_{i=1}^{m} P_i \frac{\partial x_i}{\partial P_k} + x_k = 0, \quad k = 1, \ldots m \qquad \ldots\ldots(28)$$

Together with condition (14), this yields

$$\sum_{i=1}^{m} t_i \frac{\partial x_i}{\partial P_k} = \nu \left(\sum_{i=1}^{m} P_i \frac{\partial x_i}{\partial P_k} + \frac{\partial x_o}{\partial P_k} \right)$$

which can be finally written as

$$\sum_{i=1}^{m} (t_i - \nu P_i) \frac{\partial x_i}{\partial P_k} = \nu \frac{\partial x_o}{\partial P_k}, \quad k = 1, \ldots, m. \qquad \ldots\ldots(29)$$

If the matrix $\left(\frac{\partial x_i}{\partial P_k}\right)$ is non-singular, and its inverse is written (A_{ik}), we have the tax formula, $t_i = \left(P_i + \sum_{k=1}^{m} \frac{\partial x_o}{\partial P_k} \cdot A_{ik}\right)$ ......(29)

In general, this tax structure is not proportional. We cannot conclude from (29) that uniform taxation is optimal. In the special case when $\frac{\partial x_o}{\partial P_k} = 0$, $k = 1, 2, \ldots, m$, it becomes optimal. Thus, if labour is perfectly inelastic in supply, not only with respect to its own but also to all consumer goods prices, then uniform taxation is optimal. This special case is in confirmity with the conventional wisdom that if there exists a commodity which is perfectly inelastic in demand, then it is an "ideal" object of taxation from the efficiency point of view. In this special case we have barred ourselves from taxing labour by choosing it as the numeraire, but by imposing a proportional taxes on all consumer goods, we are, in fact, able to achieve the same result. The point to be noticed is that there is a change in the relative prices of labour and consumer goods, but within the set of consumer goods all relative prices are unchanged.

Choosing labour as an untaxed numeraire good, it is seen that uniform taxation is optimal when labour is perfectly inelastic in supply. However, perfect inelasticity is not a necessary condition for uniform taxation to be optimal. The question naturally arises if there are other conditions of general economic interest which lead to same result. This question has been discussed by Atkinson and Stiglitz.[15] Sandmo has also discussed this question in context of his basic model as follows:[16]

Going back to equation (28), if $\frac{\partial x_o}{\partial P_k} = \alpha x_k$ where α is some function independent of k, then ... ...(30)

$$\sum_{i=1}^{m} P_i \frac{\partial x_i}{\partial P_k} = -(1+\alpha)\, x_k \qquad \text{... ... (31)}$$

Substituting the value of x_k from (31) into (14), we get

$$\sum_{i=1}^{m} \left(t_i - \frac{\nu}{1+\alpha} P_i\right) \frac{\partial x_i}{\partial P_k} = 0\ , \quad k = 1, \ldots, m \qquad \text{... ...(32)}$$

Thus, in this special case also, a proportional tax is optimal. The equation (32) corresponds to equation (14), and therefore a proportional tax structure provides a solution to these equations. Moreover, if the matrix, $\frac{\partial x_i}{\partial P_k}$, is non-singular, the solution is unique.

Expanding in terms of Slutsky coefficients, we have

$$\frac{\partial x_o}{\partial P_k} = -x_k \frac{\partial x_o}{\partial I} + S_{ok}\ , \quad k = 1, \ldots, m \qquad \text{... ... (33)}$$

where S_{ok} are the substitution coefficients and $\frac{\partial x_o}{\partial I}$ income effect which is here evaluated at I = 0.

Since the income effect is in fact proportional to x_k, one condition for (30) to be true is that the cross-substitution coefficients, S_{ok}, must vanish. This implies that a proportional taxation will be optimal if there are no relationships of complementarity or substitutability between labour and consumer goods.

It is already seen that the uniform taxation, i.e., taxation of all commodities including labour at equal rates, has no obvious claim to be considered as optimal. Yet in the example shown in Figure II it did after all turn out to be optimal. In this case, the conditions are (i) the income elasticities for both taxed goods are equal, and (ii) the indifference map is homothetic. However, the analysis is incomplete because the indifference map must be understood as drawn for a fixed supply of labour, while this supply will in fact change with structure of prices.

But if the indifference map is in effect invariant to the changes in the labour supply, the analysis is clearly valid. This implies that there should be another assumption of utility separability between labour and consumer goods. Under these assumptions a uniform taxation is optimal.

It is also seen that there are some interesting special cases in which uniform taxation is optimal. Still these cases are special enough to justify the conclusion that proportionality will be the exception rather than rule if an optimal tax structure is to be chosen on the basis of pure efficiency criterion.

INVERSE ELASTICITY RULES

The inverse elasticity rules discussed in Section I are based on partial consumer surplus analysis. On the basis of general equilibrium analysis, these rules can be easily derived as special case of the basic model of optimal commodity taxation.

If the demand functions (9) are independent of each other all cross-derivatives of the demand functions vanish as between the taxed goods. Thus,

$$\frac{\partial x_i}{\partial P_k} = 0, \; i \neq k \qquad \ldots \ldots (34)$$

Then the conditions (14) take the simple form:

$$t_k \frac{\partial x_k}{\partial P_k} = -\left(\frac{\mu-\lambda}{\mu}\right) x_k = -\nu x_k$$

or,

$$\frac{t_k}{P_k} = \frac{\nu}{\varepsilon_{kk}}, \quad k = 1,2, \ldots m. \qquad \ldots \ldots (35)$$

where, $\varepsilon_{kk} = -\frac{P_k}{x_k} \frac{\partial x_k}{\partial P_k}$, the (direct) price elasticity of demand for good k.

The equation (35) is the well-known elasticity formula for commodity taxation. It says that the tax rates should be highest on the commodities with the lowest price elasticities of demand in order to minimize the deviations from the non-distortive pre-tax optimality.

Commenting on the inverse elasticity rule, Dixit[17] argues that this rule is not valid if all the commodities are taxable, since with independent demands all price elasticities must be equal to minus one. But correcting the Dixit result, Sandmo[18] says that it is still valid since no assumption is made about the derivatives $\frac{\partial x_o}{\partial P_k}$. If, in addition to the demand independence assumption, one assumes $\frac{\partial x_o}{\partial P_k} = 0$, for all k, we are back to a proportional structure, but this is simply a special case of the more general result derived in the previous section.

Nevertheless, elasticity formulae become very complex in the general model and provide little intuitive insight into the optimum structure of commodity taxation. One special case which is worthy to examine is a three-good economy involving labour and two taxed consumer goods. This case was first examined by Corlett and Hague[19] discussing the direction of movement away from proportional taxation that would increase utility. Later Diamond and Mirrlees[20] as well as Sandmo[21] also discussed this case to illustrate their models.

In the three-good case, with good 0 untaxed, the first-order conditions (18) reduce to the following two equations:

$$t_1S_{11} + t_2S_{12} = -Kx_1$$
$$t_1S_{21} + t_2S_{22} = -Kx_2 \qquad \ldots\ \ldots(36)$$

where $K = \nu - \sum_{i=1}^{2} t_i \frac{\partial x_i}{\partial I}$, an expression independent of k.

These two simultaneous equations can be solved by Cramer's rule to yield

$$t_1 = -K\frac{x_1S_{22} - x_2S_{12}}{S_{11}S_{22} - S_{12}^2}$$
$$t_2 = -K\frac{x_2S_{11} - x_1S_{21}}{S_{11}S_{22} - S_{12}^2} \qquad \ldots\ \ldots(37)$$

where the denominator is positive by the properties of Slutsky matrix.

The solutions (37) can be expressed in terms of ad valorem tax rates and compensated elasticities as follows:

Let θ_1 and θ_2 denote the ad valorem taxes on good 1 and good 2 respectively, so that

$$\theta_1 = \frac{t_1}{P_1}, \qquad \ldots\ \ldots(38)$$
$$\theta_2 = \frac{t_2}{P_2}$$

Also, let the elasticity of compensated demand be defined by

$$\sigma_{ki} = S_{ki}\,(P_i/x_i)\ ,\ i, k = 1,2. \qquad \ldots\ \ldots(39)$$

Then, the solutions (37) can be expressed as

$$\theta_1 = - K \frac{1}{S_{11}S_{22}-S_{12}^2} \frac{x_1x_2}{P_1P_2} (\sigma_{22} - \sigma_{12}) = - K^1 (\sigma_{22} - \sigma_{12})$$

... ...(40)

$$\theta_2 = - K \frac{1}{S_{11}S_{22}-S_{12}^2} \frac{x_1x_2}{P_1P_2} (\sigma_{11} - \sigma_{21}) = - K^1 (\sigma_{11} - \sigma_{21})$$

Now, by the adding-up properties of compensated elasticities, we have,

$$\sigma_{10} + \sigma_{11} + \sigma_{12} = 0$$

... ...(41)

$$\sigma_{20} + \sigma_{21} + \sigma_{22} = 0$$

Finally, by substituting for σ_{12} and σ_{21} in the solutions for the tax rates (40), we get

$$\theta_1 = - K^1 (\sigma_{11} + \sigma_{22} + \sigma_{10})$$

... ... (42)

$$\theta_2 = - K^1 (\sigma_{11} + \sigma_{22} + \sigma_{20})$$

Since labour supply is being measured negatively in this model, K^1 has the same sign as net government revenue. For definiteness, suppose that government revenue is positive so that $K^1 >> 0$. Then, from (42), it follows that

$$\theta_1 \gtreqless \theta_2 \quad \text{according as } \sigma_{10} \quad \sigma_{20}$$ ... ...(43)

Thus, the consumer good which has lower compensated cross-elasticity of demand with the price of labour should be taxed at the higher rate. This means that a consumer good which is a substitute for labour should be taxed at a higher rate than one which is complementary with labour. The economic rationale of this result is easily understood that since we have barred ourselves from taxing leisure, we can achieve our target by taxing the goods which are complementary with leasure. "It is important, however, to emphasize that it has nothing to do with leisure per se. The general principle is that if we have one untaxed good, we should tax more heavily that good most complementary with it, since it is a way of indirectly "taxing" the untaxed good. It just happens that we are here assuming that leisure is untaxed."[22]

PRODUCTION EFFICIENCY

The basic model described in the previous sections proceeds on the assumption of fixed producer prices, and thus neglects the possibility that a tax structure might distort efficiency by causing misallocation of resources in production. This assumption produces a set of optimum tax rules which does not consider the production and supply conditions. Diamond and Mirrlees[12] as well as Dixit[14] have shown that the constant producer prices assunption is equivalent in terms of its implications to the more general assumption of constant returns to scale in production. Nevertheless, these tax rules have to be modified if there is no constant returns to scale in production.

Another important question to be considered is that of production efficiency. Efficiency will be present if the producer prices are equal to marginal costs in production. Constant producer prices, therefore, imply constant marginal costs. But, the marginal costs under constant returns to scale are constant only in a partial equilibrium sense, not in a general equilibrium sense. However, Diamond and Mirrless[13] have shown that the presence of optimal commodity taxes implies the desirability of production efficiency, although a full Pareto optimum in the economy is not achieved.

It seems natural, analytically, to approach the production efficienc issue in optimal commodity taxation by introducing a production constrain into the basic model and determining the producer prices by profit-maximization technique. This is the approach Dixit[14] applied to extend his model to accommodate production conditions.

Suppose one of the (m+1) commodities is not subject to taxation. There is no loss of generality in this assumption, since the prices can be normalized to have zero tax on one commodity when the production is considered. For convenience, the good 0, i.e., labour, is supposed not subject to taxation.

Let $y = (y_o, y_1, \ldots \ldots y_m)$ be the vector of commodities supplied, inputs being treated as negative outputs, then the production constraint is given by

$$Y(y) = 0, \text{ or, equivalently,}$$

$$y_o = f(y_1, y_2, \ldots \ldots y_m) \qquad \ldots \ldots (44)$$

For the economy to be in equilibrium, the supply of each good must be equal to the demand for that good for some price. In vector notation,

$$y = x(P). \qquad \ldots \ldots (45)$$

where $x = (x_o, x_1, \ldots \ldots x_m)$ is the vector of consumer demand and $P = (P_o, P_1, \ldots \ldots P_m)$ is the vector of consumer prices. Therefore, the equation (44) becomes, in equilibrium,

$$x_o = g(x_1, x_2, \ldots \ldots x_m) \qquad \ldots \ldots (46)$$

Now, if the good 0 (i.e., labour) is chosen as numéraire, the profit (i.e., the net value of the production) is given by

$$x_o + \sum_{i=1}^{m} p_i x_i \qquad \ldots \ldots (47)$$

where $p = (p_1, p_2, \dots \dots p_m)$ is the vector of producer prices.

In what follows, it is assumed that no price p_i is negative, so that the producers lose nothing by limiting themselves to technically efficient net productions. Obviously, it is also assumed that the price vector p is not identically zero.

Adopting the assumptions of profit maximization and perfect competition, and using a production function representing the technical constraints, we can easily determine equilibrium for the producers. The producers try to maximize their profit (47) subject to the production constraint (46). The necessary first-order conditions for a vector x to be a solution imply the existence of a Lagrange multiplier λ' such that[23]

$$p_o = \lambda'$$
$$p_i = -\lambda' \frac{\partial g}{\partial x_i}(x_1, x_2, \dots \dots x_m) \qquad \dots \dots (48)$$

where $\frac{\partial g}{\partial x_i}$ is negative, since inputs are treated as negative outputs.

Because good 0 is chosen as numéraire, Lagrange multiplier λ' is unity. Therefore, marginal cost of good i, $-\frac{\partial g}{\partial x_i}$, is equal to its price p_i. Thus, there is production efficiency.

Now suppose the government wants to raise a fixed amount of revenue, T, by taxing the commodities x_i's. It would be a more natural approach to use the taxes which the government actually controls as decision variables. However, once we have determined the optimal P and p vectors we have determined the optimal taxes. Since (48) gives p_i as functions of the x_i's, and (9) gives x_i as functions of the P_i's, it is convenient to take P_i as the independent variables.

Suppose the government desires to maximize social welfare function,

$$U = U(g(x_1, x_2, \dots, x_m), x_1, x_2, \dots, x_m) \qquad \dots \dots (6')$$

subject to its revenue constraint,

$$\sum_{i=1}^{m} (P_i - p_i)x_i = T. \qquad \dots \dots (1')$$

We can formulate this problem in terms of a Lagrange function,

$$L_1 = U(g(x_1, x_2, \dots, x_m), x_1, x_2, \dots, x_m) + \mu_1 \sum_{i=1}^{m} (P_i - p_i)x_i - T) \dots (49)$$

and we obtain the necessary conditions for a constrained maximum of U by setting the partial derivatives of L_1 with respect to the consumer prices equal to zero:

$$\{\frac{\partial U}{\partial g}\cdot\frac{\partial g}{\partial P_k}+\sum_{i=1}^{m}\frac{\partial U}{\partial x_i}\cdot\frac{\partial x_i}{\partial P_k}\}+\mu_1\{x_k+\sum_{i=1}^{m}(P_i-p_i)\frac{\partial x_i}{\partial P_k}-\sum_{i=1}^{m}\sum_{h=1}^{m}x_i\frac{\partial p_i}{\partial x_h}\frac{\partial x_h}{\partial P_k}\}=0$$

where h, k = 1,2, ..., m. ... ... (50)

These conditions can be simplified by taking into consideration of the optimizing behaviour of consumers, who maximize their utility function (6') subject to their budget constraint (1). The optimum conditions for consumers are,

$$\frac{\partial U}{\partial g}=\lambda_1 P_o=\lambda_1\quad(\because p_o=1)$$

$$\frac{\partial U}{\partial x_i}=\lambda_1 P_i\;,\; i=1,2,\ldots,m. \qquad \ldots\ \ldots\ (51)$$

Therefore, $\frac{\partial U}{\partial g}\cdot\frac{\partial g}{\partial P_k}+\sum_{i=1}^{m}\frac{\partial U}{\partial x_i}\cdot\frac{\partial x_i}{\partial P_k}$

$$=\lambda_1\sum_{i=1}^{m}\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial P_k}+\lambda_1\sum_{i=1}^{m}P_i\frac{\partial x_i}{\partial P_k}\;,\ \text{since } x_o=g\,(x_1,x_2,\ldots x_m)$$

$$=\lambda_1\sum_{i=1}^{m}(P_i-p_i)\frac{\partial x_i}{\partial P_k}\;,\quad\text{using (48)}$$

This enables us to write the conditions (50) as

$$\frac{\lambda_1+\mu_1}{\mu_1}\sum_{i=1}^{m}(P_i-p_i)\frac{\partial x_i}{\partial P_k}=\sum_{i=1}^{m}\sum_{i=1}^{m}x_i\frac{\partial p_i}{\partial x_h}\frac{\partial x_h}{\partial P_k}-x_k$$

and finally as, interchanging the dummy indices i and h in the double summation,

$$\sum_{i=1}^{m}\{t_i-\nu_1\sum_{h=1}^{m}x_h\frac{\partial p_h}{\partial x_i}\}\frac{\partial x_i}{\partial P_k}=-\nu_1 x_k \qquad \ldots\ \ldots\ (52)$$

where h, k = 1,2, ..., m ; and $\nu_1=\frac{\mu_1}{\lambda_1+\mu_1}$.

Since there is an untaxed commodity 0, viz. labour, substituting the value of x_k from (13) into (52), noting that $P_o = 1$, we get,

$$\sum_{i=1}^{m}(t_i-\nu_1\sum_{h=1}^{m}x_h\cdot\frac{\partial p_h}{\partial x_i})\frac{\partial x_i}{\partial P_k}=\nu_1(\frac{\partial x_o}{\partial P_k}+\sum_{i=1}^{m}P_i\frac{\partial x_i}{\partial P_k})$$

$$\text{or, } \sum_{i=1}^{m} \{t_i - \nu_1 (P_i + \sum_{i=1}^{m} x_h . \frac{\partial p_h}{\partial x_i})\} \frac{\partial x_i}{\partial P_k} = \nu_1 \frac{\partial x_o}{\partial P_k} \quad \dots \dots (53)$$

If the matrix $(\frac{\partial x_i}{\partial P_k})$ is non-singular and its inverse is written (A_{ik}), we have the tax formula

$$t_i = \nu_1 (P_i + \sum_{k=1}^{m} \frac{\partial x_o}{\partial P_k} . A_{ik} + \sum_{h=1}^{m} x_h . \frac{\partial p_h}{\partial x_i}) \quad \dots \dots (54)$$

The tax formula (54) is different, apart from the special case of one untaxed commodity, from the tax formula (29') in that it involves differential coefficients of producer prices with respect to x_i's. Thus, the production can affect the optimum tax structure under general conditions subject only to an interior maximum.

This difference vanishes, however, if there is constant returns to scale in production. To assume constant returns to scale is not the same thing as assuming infinitely elastic supply curves: constant returns to scale are perfectly compatible with diminishing returns to one factor of production, and in that case, (48) shows that the supply curve depicting p_i against x_i has a positive slope.

If g $(x_1, x_2, \dots, x_m)$ is a homogeneous function of degree one in all the x_i's, then each $\frac{\partial g}{\partial x_i}$ is homogeneous of degree zero in them,

and
$$\sum_{h=1}^{m} x_h . \frac{\partial p_h}{\partial x_i} = - \sum_{h=1}^{m} x_h . \frac{\partial^2 g}{\partial x_h . \partial x_i} \text{, using (48)}$$

$$= - \sum_{h=1}^{m} x_h . \frac{\partial^2 g}{\partial x_i . \partial x_h}$$

$$= \sum_{h=1}^{m} x_h . \frac{\partial p_i}{\partial x_h}$$

$$= 0 \text{, by Euler's theorem.}$$

In this case, the tax formula (54) reduces to (29'). This special case of Dixit analysis confirms the result obtained by Diamond and Mirrlees.

FOOTNOTES

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1. Alfred Marshall, Principles of Economics, Eight Edition (London: 1938), p. 467.

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