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THE COBB DOUGLAS FUNCTION AND ITS EXTENSION

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Beginning with the Cobb Douglas production function, this paper analyses several production models which are derivable from the Cobb Douglas function by means of simple changes in the latter. It is maintained that theoretical justification, empirical attractiveness, ability to extract meaningful structural relationship from the data, the use of practical and measurable variables and the ease of statistical estimation are some of the basic requirements of any production model. The paper begins with an elementary introduction to the subject.

To the economist, the production function is a tool for the explanation of decisions already made. The manager attempts to select the best production decision by elimination from a set of alternatives for actual implementation. The engineer maximises output for a given set of inputs through his choice of technology. The profit maximising firm subjects the production function to restrictions regarding rewards to inputs for their contribution to the output and suitably combines technological considerations with economic requirements.

The production function, as an embodiment of technological constraints imposed on economic decisions, is not supposed to includs explicitly economic variables like interest or prices. The form of the relationship between outputs and inputs is not based on economic decisions but the behavioural and organisational aspects are not excluded; Monga (1979). In spite of controversies about various matters like theoretical justification, inconsistencies in

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results, aggregation problems etc. associated with the production function, soms production models in common use have done reasonably well on purely empirical grounds. They have been successfully employed at various levels of aggregation and for different kinds of data. At the same time a search continously goes on to find a better expression for a production model which is technically and economically meaningful and satisfies certain basic requirements. Simple as wel as very complex models have been tried, theoretically and empirically with plenty of applications to industrial data. It is realised that a complicated expression does not necessarily provide a better model, a simpler form may sometimes be more useful. Theoretical justification so far as possible, empirical attractiveness, ability to extract meaningful structural relationship from the data, inclusion of such variables as are available in practice and are measurable and the ease of the estimation of parameters are some of the basic requirements of any production function model.

So far as the study of a variety of production function forms and their use in the literature is concerned, the situation is not very happy. There are some interesting surveys of production function studies like those of Walter (1963), Hildebrand and Liu (1965) and Nerlove (1967). It seems necessary now to go for a more practical and perhaps more ambitious survey on different lines based on the way different forms evolved along with a consideration of their connecting links and the factors entering into various production models. Perhaps a number of forms could be derived from a few generalised production models but that would conceal the essence of the development of the idea of the production function. Such an approach or a catalogue of forms cannot help us in understanding the evolution and spreading of the production function idea. The scope of this paper is limited to the analysis of some of the extensions of the Cobb Douglas production function only. This has been done, one, because of limitations of space, two, because other forms have been treated independently in other papers (yet to be published) and three, because the nature of analysis for other models of production function differs considerably from that of Cobb Douglas function which is treated here.

The overall plan of this work covers and analyses a large number of forms of production functions, developed during the last two decades, and presents them in a form which may be found useful and handly in as much as that should make the choice of a suitable form easier in empirical work. Adjustments and improvements in the available forms, depending on requirements, should also be more convenient by this procedure. The subject has been highly scattered and developed at an uneven pace, concentrating on some aspect or the other of the problem at different periods of time depending on requirements. However on a careful observation of the undercurrent, some systematic and continuous development cannot fail to be noticed. A mere categorisation of some forms is neither feasible nor meaningful. What is required is a study of the development of the idea, without sacrificing the value of individual attempts which though sometimes trivial, made some contribution to the subject. A quick derivation from some mathematically general form, of all the particular forms will suppress the role of individual forms and make them look trivial. At the same time the possibility of further extensions from simpler forms in different directions and for different purposes cannot be ruled out. Removal of certain restrictions, use of cost functions, introduction of or removal of some explanatory factors, incorporation in a suitable form of experience gained from empirical studies and introduction of certain assumptions about the nature of the parameters involved are some of the strategies used to bring about an improvement of existing forms. It may be found more realistic in some cases to incorporate in the production function such assumptions as variability of elasticity of substitution, returns to scale and marginal products.

The treatment of the various models in this paper will, perforce, have to be rather brief but attempt will be made to leave out nothing of importance.

The Definition of the Neoclassical Production Function and Related Concepts

The neoclassical production function is a mathematical statement expressing the technological relationship between the output of a process and the inputs entering into the process with possibilities of substitution. Let $X=(X_1, .. , X_n)$ be the vector of n inputs X_1 $X_2, ..., X_n$ where each $X_i \geqslant 0$. Then, to each point in the input space there is a unique non. negative output point. The general production function for a single output Q, produced from n variable inputs, may be written.

$$Q = F(X_1, X_2, ..., X_n)$$

This function is assumed to be single valued and continuously differentiable.

There exists an economic region which is a subset of the input space in which output dose not decrease as input increases. For any two vector point x_i) and x_i ($\geqslant x_i$) in the economic region we have $F(x_j) \geqslant F(x_i)$ which implies that the first partial derivatives or marginal products are nonnegative.

$$\partial F / \partial x \geqslant 0 \qquad i=1,2,..,n$$

The law of diminishing returns requiries $\delta^2 F/\delta x_i^2 < O$, $i=1,2,..,n$.

In a convex subset of the economic region the Hessian his negative definite where

$$h = \begin{vmatrix} \frac{\delta^2 F}{\delta X_1^2} & \frac{\delta^2 F}{\delta X_1 \delta X_2} \cdots & \frac{\delta^2 F}{\delta X_1 \delta Xn} \\ \cdot & \cdots & \cdot \\ \frac{\delta^2 F}{\delta Xn \delta X_1} & \frac{\delta^2 F}{\delta Xn \delta X_2} \cdots & \frac{\delta^2 F}{\delta Xn^2} \end{vmatrix}$$

Further, $F(vx) \gtreqless vF(X)$ according as the production function exhibits increasing constant or decreasing returns to scale. Also, the point returns to scale $E(X) = \sum E_i(X)$ where the elasticity of output with respect to the i th input is

$$E_i(X) = (X_i / F)\, \delta F/\delta X_i \quad i=1,2,...n.$$

The elasticity of substitution between two inputs, X_i, X_j, with other inputs held constant, is given by

$$\delta ij = \frac{d\ln \frac{Xi}{Xj}}{d\ln \frac{(\delta F/\delta F)}{(\delta Xi/\delta Xj)}}$$

which is also the curvature of the isoquants. Since, along an isoquant,

$\sum_i^n \frac{\delta F}{\delta X_i} dX_j = 0$, we have, in the two input case

$$\frac{\delta F}{\delta X_i} dX_i + \frac{\delta F}{\delta X_j} dX_j = O$$

$$\therefore \frac{dX_i}{dX_j} = -\frac{\delta F/\delta X_j}{\delta F/\delta X} = \frac{MP_j}{MP_j}$$

where MP_i, MP_j stand for the marginal products of X_i, X_j respectively. Hence

$$\sigma_{ij} = \frac{d\ln x_i / x_j}{d\ln(-dx_i/dx_j)}$$

The feasible region of production space is the closed half space difind by

$$Q \leqslant F(X_1, X_2, ..., Xn).$$

This is merely a description of the boundary of production choices relating Q with X_1, X_2, ..., X_n

Elasticity of Substitution

In the two input (x_1, x_2) case, the marginal rate of technical substitution (MRS) may be written

$$MRS = s = dx_2 / dx_1 = F_1 / F_2$$

where the suffixes to F denote appropriate partial derivatives.

Writing $x = X_2 / X_1$, we have, for variation along an isoquant, the elasticity of substitution, in the two input case, corresponding to substitutions for a constant output level, defined by

$$\sigma = \frac{dX/X}{ds/s},$$

It can be shown that

$$\sigma = \frac{F_1 \; F_2 \, (X_1 \; F_1 + X_2 \; F_2)}{X_1 \; X_2 \, (F_{ii} \; F_2^2 - 2F_{12} \; F_1 \; F2 + F_{22} \; F_1^2)}$$

or

$$\sigma = \frac{S}{X1 \; X_2} \quad \frac{X_{is} + X_2}{d^2 X_2 / dX_1^2}$$

Which is nonnegative and lies between zero and infinity. Also σ is inversely proportional to changes in the isoquant slope.

For a homogeneous production function of degree one, it can be shown that $\sigma = F_1 \, F2/F \; F_{12}$

If we write $y = Q/X_1$, $x = X_2 / X_1$, the formula for elasticity of substitution, instead of remaining a partial differential equation, can be written as a nonlinear differential equation

$$\sigma = - \frac{v' \, (y - x \, v')}{x \, y \, y''}$$

This is an extensively used relation in production function studies.

For the n-input production function $Q = F(X_1, X_2, ..., X_n)$ homogeneous of degree one, the bordered Hessian may be written

$$H = \begin{vmatrix} 0 & F_1 & F_2 & \dots & F_n \\ F_1 & F_{11} & F_{12} & \dots & F_{1n} \\ F_2 & F_{21} & F_{22} & \dots & F_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ F_n & F_{n1} & F_{n2} & & F_{nn} \end{vmatrix}$$

Where the suffixes to F show appropriate partial derivatives.

Denoting the cofactor of F_{ij} by H_{ij} we have the Allen partial elasticity of substitution between X_i, X_j given by

$$\sigma_{ij} = \frac{\Sigma X_i \; F_i}{X_i \; X_j} \; \frac{H_{ij}}{H} = \sigma_{ji} \text{ (symmetry), } i / j$$

Using Euler's thorem,

$$\sigma_{ij} \frac{Q}{X_i \; X_j} - \frac{Hij}{H}$$

from which the result for the two input case follows easily.

Some Basic Modle

The simplest functional form for a production model is the linear form with two or more inputs. The n input model $Q = a + \sum_{i=1}^{n} b_i \; X_i$ is a production function if Q and Xi are non-negative and $\delta Q / \delta X_i = b_i > O$. The elasticity of substitution is infinity and the output elasticity is unity. This production model is not used in practice. It is possible to think of a firm having only some discrete choices in the matter of inputs, their quantities and use. Based on this assumption of constancy of engineering and technological factors that determine the relationship between inputs we have the Leontief input-output function which is also linear. In the two input case, let c_1, c_2 stand for the amounts X_1, X_2 needed to product one unit of output. The Leontief function may be written

$$Q = \min \; (X_1 / c_1, \; X_2 / c_2)$$

$$\text{or } X_1 \geqslant c_1 \; Q, \qquad X_2 \geqslant c_2 \; Q$$

Because the two inputs move together the marginal products remain undefined. The elasticity of substitution is 0. The output elasticity is unity provided $X_1 / c_1 = X_2 / c_2$ In the theory of production functions, the first major contribution was the Cobb Douglas production function which has had a long life span and is still being used on a large scale in spite of a number of other models now available.The Leontief fixed proportion case has important applications in a specialised input output framework. A commonly used form in recent years is the constant elasticity of substitution (CES) production function which is a generalisation of the Cobb Douglas function. The popularity and usefulness of these forms have given rise to a number of extensions and more complex forms which perform additional roles and are supposed to have certain desirable properties. Reference may be made to the surveys of Walters (1963), Hildebrand and Liu (1965) and Nerlove (1967).

Because of their relative simplicity and manageability, the Leontief, Cobb Douglas and the CES functions have yielded a number of useful and interesting production function studies. Recently some variable elasticity of substitution (VES) functions have also been used in empirical work. None of these functions has all the empirically desirable character-estics. Often additional qualities are brought into them at a price which consists of som simplifying and very likely, unrealistic assumptions. In any case, the mathematical representation of any form does not have much meaning unless supported by empirical results.

The Leontief function does not allow for substitution between inputs. The capital labour ratio is uniquely determined and has nothtng to do with prices. It means that for any output there is only one production process. The Cobb Douglas function allows for factor substitution but the elasticity of substitution is restricted to unity. Along an isoquant, the proportional change in inputs for a given change in input price is ratio is fixed. This is also the case with the CES function but the extent of this change is a parameter of the CES function and not fixed in advance as in the Cobb Douglas case. The VES function allows the variability along an isoquant, of the elasticity of substitution which is proportional to the input ratio. In the case of more than two inputs, we have the translog production function which is subject to a minimum number of prior restrictions and is amenable to tests of degree of returns to scale and separability.[1]

We now carry out a brief analysis of the Cobb Douglas function and some other forms which may be considered as its extensions. Harter-Carter-Hocking's (1960) Transcendental

1. The restrictions of separability and aggregation can be imposed on the translog function as testable parametric restrictions.

This is a very useful feature for empirical work.

Production Function. Vinod's (1972) Homogneous Function of Variable Degree.

Chu–Aigner–Frankel's (1970) Log Quadratic Law of Production.

Sudit's (1973) Additive Nonhomogeneous Function.

Janvry's (1972) Generalised Power Production Function.

Kmenta's (1967) CES Approximation.

Christensen–Jorgensen–Lau's (1971) Translog Function.

The Cobb Douglas Production Function

The two input Gobb Douglas production function is usually written in the form

$$Q = A K^{\alpha} L^{B}$$

where K and L stand for capital and labour respectively. Q is the output produced. The function satisfies the neoclassical requirements in that the marginal products are positive:

$$\partial Q/\partial K = \alpha Q/K > 0, \ \partial Q/\partial L = BQ/L > 0.$$

also $\partial^2 Q/\partial K^2 = \frac{\alpha(\alpha-1)Q/K}{K} < 0, \ \partial^2/\partial L^2 = \frac{B(B-1)}{L} QL < 0.$

α and B are output elasticities. The returns to scale are given by $\alpha + B$. A is the efficiency coefficient and α/B is the degree of input intensity. Writing $X = K/L$, the elasticity of substitution is given by

$$\sigma = d \ln X/d \ln s = 1 \text{ since } s = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{BK}{\alpha L}.$$

Written in the form $Q = A^{K^{\alpha} L'^{B}}$ the Cobb Douglas function involves only technical variables. But as it is difficult to measure the physical output Q in suitable units, Q is replaced by Y, total value of output, or by V, value added. It is difficult to express K, capital assets, in suitable physical terms and therefore, even K has to be used in value terms. As for the problem of lack of homogeneity, it is common to all inputs.

The use of money values in place of physical quantities introduces an economic element into the purely technical relationship of the production function. This is unavoidable in practice and may be the cause of some differences in conclusions drawn from empirical production function studies based on the assumption of a purely technical production relation.

If we write α_i for the output elasticity of X_i, the n–input Cobb Douglas function may be written

$$Q = A X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n}$$

Also $\partial Q/\partial X_i = \alpha_i Q/X_i > 0$ and $\partial^2 Q/\partial X_i^2 = \dfrac{\alpha_i(\alpha_i - 1) Q}{X_i} < 0$,

for $i = 1, 2, \ldots\ldots\ldots, n$. The returns to scale is given by $\sum \alpha_i$ and $\sigma_{ij} = 1$, i/j.

With the constraint $\alpha + B = 1$, the Cobb Douglas function with two inputs K and L may be written as a productivity relation between average productivity and capital intensity. The assumption of constant returns to scale made here may or may not be true. Dividing the unrestricted Cobb Douglas equation throughout by L, we have

$$Q/L = A K^{\alpha}/L^{1-B}$$

$$= A(K/L)^{\alpha} \text{ since } \alpha + B = 1$$

To test the hypothesis of constant returns to scale, that is, to test $\alpha + B = 1$, the relation may be written

$$Q/L = A (K/L)^{\alpha} L^{\alpha + B - 1}$$

The significance of the coefficient of L can be used to verify the hypothesis that $\alpha + B$ adds up to unity.

The Extension Procedure

We now introduce additional explanatory factors into the Cobb Douglas relation. K/L is a good explanatory factor and may be introduced into the Cobb Douglas function. But since it leaves the latter unaltered in form, a term like $(\ln K/L)^2$ may be used in the log linear Cobb Douglas relation. This gives rise to a new productivity relation which happens to coincide with an approximation by Taylor's expansion of the CES as well as the VES functions. The CES approximation is usually called Kmenta approximation.

From the procedur just mentioned it can be seen that the introduction of an additional factor into an existing form gives rise to a new form of production function. The resulting form may not necessarily continue to retain the original properties like those of homogeneity or constancy of returns to scale or unitary elasticity of substitution that may be present in the basic form with which we may start.

If to the right hand side of the Cobb Douglas function for n inputs,

$$Q = A X_1^{\alpha_1} X2^{\alpha_2} \dots \dots X_n^{\alpha_n}$$

multiplicative exponents of inputs are introduced we get an earlier form of transcendental production function given by Harter, Carter and Hocking (1960). There are several nonhomogeneous variations of the Cobb Douglas function. Kmenta approximation obtained by adding $(\ln K/L)^2$ as an additional explanatory factor to the Cobb Douglas log linear relation results in a nonhomogeneous function. Another nonhomogeneous function is that of Vinod (1972) which is obtained by adding (ln K.ln L) in the two input Cobb Douglas loglinear relation. In other words, it is obtained by making each input exponent a linear function of the other input in the two input Cobb Douglas function. If instead, the exponents are made linear functions of input ratios, we get Sudit's (1973) homogeneous function which results in the addition of three terms to the Cobb Douglas linear relation, viz $(\ln K)^2$, $(\ln L)^2$ and (ln K. ln L). If only two ter ms, viz.,L lnK and KlnL are added the resulting production function is nonhomogeneous. Other nonhomogeneous functions which may be considered as some kinds of extensions of the Cobb Douglas function are those of chu–Aigner–Frankel (1970), Janvry (1972) and the by now quite famous translog production function of Christensen, Jorgensen and Lau (1971).

Some Extensions of the Cobb Douglas Function

One of the earlier extensions of the Cobb Douglas function was obtained by using exponents of inputs as additional multiplicative factors in the Cobb Douglas form. Harter, Carter and Hocking (1960) called it the transcendental production function. It may be written

$$Q = A X_1^{\alpha 1} e^{\beta 1 X1} \dots \dots \dots X_n^{\alpha n} e^{\beta_n X_n}.$$

In the two input case the marginal rate of technical substitution is given by

$$S = \frac{\alpha_2 + B_2 X_2}{\alpha_1 + B_1 X_1} \quad \frac{X_1}{X_2}$$

The function exhibits nonconstant elasticity of substitution.

It has also the characteristic of allowing marginal products to rise before eventually falling.

The elasticities of production are given by

$$E_1 = \partial \ln Q / \partial \ln X_1 = \alpha_1 + B_1 X_1$$

$$E_1 = \partial \ln Q / \partial \ln X_2 = \alpha_2 + B_2 X_2$$

So that the scale of production elasticity is

$$E = \alpha_1 + \alpha_2 + B_1 X_1 + B_2 X_2$$

The Harter, Carter and Hocking relation adds a linear function of inputs to the Cobb Douglas function written in the log linear form

$$\ln Q = \ln A + \Sigma \alpha_i \ln X_i + \Sigma B_i X_i$$

The nonhomogeneous production function of Vinod (1972) provides an extension of the Cobb Douglas function by substituting α and B (of $Q = AK^{\alpha} L^{B}$) by linear functions of inputs. α is replaced by a linear function of L and B by a linear function of K. Thus

$$Q = A K^{(a_1 + c_1 \ln L)} L^{(a_2 + c_2 \ln K)}$$

This function adds an interactive terms to the linear Cobb Douglas relation which with $a_3 = c_1 + c_2$ may be written $\ln Q = \ln A + a_1 \ln K + a_2 \ln L + a_3 \ln K . \ln L$

∵ a_3 is not significantly different from zero, the Cobb Douglas function is implied.

The output elasticities are given by

$$E_L = a_2 + a_3 \ln K, \; E_K = a_1 + a_3 \ln L$$

and the scale elasticity

$$E = E_L + E_K = a_1 + a_2 + a_3 \ln LK$$

which is variable and dependent on input levels.

If in the elasticity of substitution expression, Allen (1938, p. 342), we substitute

$$F_K = (a_1 + a_3 \ln L)\, Q/K, \quad F_L = (a_2 + a_3 \ln K)\, Q/L$$

$$F_{LL} = Q E_L (E_L - 1)/L^2, \quad F_{KK} = Q E_K (E_K - 1)/K^2$$

$$F_{KL} = (E_K E_L Q/L + a_3 Q/L)/K = (E_K E_L + a_3)\, Q/KL$$

we have

$$\sigma = \frac{-F_K F_L (KF_K + LF_L)/KL}{F_{KK} F_L^2 - 2F_{KL} F_K F_L + F_{LL} F_K^2}$$

$$= \frac{E_L + E_K}{E_L + E_K + 2a_3}$$

$$= \frac{a_1 + a_2 + a_3 \ln KL}{a_1 + a_2 + a_3 (2 + \ln KL)}$$

which is less than unity if $E_L + E_K \lesseqgtr 0$ and $a_3 \lesseqgtr 0$.

The function is reasonably nonrestrictive and is a natural generalisation of the Cobb Douglas function.

If the α and B of the Cobb Douglas function are replaced by log linear functions of input ratios, we have Sudit's (1973) Homogeneous production function of variable degree with variable & elasticity of substitution and returns to scale:

$$Q = A\, K^{a_1 + c_1 \ln K/L}\, L^{a_2 + c_2 \ln L/K}$$

It is homogeneous of degree $a_1 + a_2 + c_1 \ln K/L$, and implies that different production techniques as reflected by different input ratios generate different scale factors. It reduces to the Cobb Douglas from with $c_1 = O = c_2$.

For estimation purposes it may be written

$$\ln Q = \ln A + a_1 \ln K + a_2 \ln L = a_{12} \ln K \ln L$$

$$- c_1 (\ln K)^2 - c_2 (\ln L)^2 \text{ where } a_{12} = c_1 + c_2$$

$$\text{or } \ln Q = \ln A + a_1 \ln K + a_2 \ln L + c_1 (\ln K \ln L - (\ln K)^{-2})$$

$$+ c_2 (\ln K \ \ln L - (\ln L)^2)$$

The output elasticities are given by

$$E_K = a_1 + (c_1 + c_2) \ln L - 2c_1 (\ln K)/K$$

$$E_L = a_2 + (c_1 + c_2) \ln K - 2c_2 (\ln L)/L$$

The elasticity of substitution

$$\sigma = \frac{E_L + E_K}{E_K + E_L + 2c_1 (\ln L - 1) \dfrac{LE_L - EK}{LE_L} + 2c_2 (\ln K - 1) \dfrac{KE_K - E_L}{KE_K}}$$

$= 1$ if c_1 and c_2 are zero.

Although it is a more flexible form than the Cobb Douglas function and has variable elasticity of substitution and returns to scale, it may suffer from the effect of multicollinearity if K and L happen to be highly collinear. The scale elasticity varies only along the isoquants. Along the expansion path, this function retains the property of homogeneity.

Sudit's (1973) additive nonhomogeneous production function (ANH) has a number of desirable properties. The function written in the general form for two inputs

$$Q = a_1 X_1 + a_2 X_2 + a_{12} X_1 \ln X_2 + a_{21} X_2 \ln X_1$$

has marginal products which are functions of the input ratio 19 and the remaining input

$$\partial Q \partial / X_1 = a_1 + a_{12} \ln X_2 + a_{21} X_2 / X_1$$

$$\partial Q / \partial X_2 = a_2 + a_{21} \ln X_1 + a_{12} X_1 / X_2$$

This implies that the abundance of a factor lowers its marginal product and the marginal cost of the other factor rises. The law of diminishing returns is thus satisfied. But the function is not necessarily restricted to diminishing returns since

$$\partial^2 Q / X \partial_1^2 = - a_{12} X_{2} / X_1^2 \text{ and } \partial^2 Q \partial / X_2^2 = - a12X_1 \; X_2^2$$

which means increasing returns from both inputs are possible for $a_{21} \; a_{12} < O$.

The shift in the marginal product of one input in response to a change in the other indicates the extent of their complementarity or competitiveness;

$$\frac{\partial^2 Q}{\partial X_1 \partial X_2} = \frac{a_{12}}{X_2} + \frac{a_{21}}{X_1}$$

The scale elasticity

$$E = E_1 + E_2 = \frac{X_1}{Q} \frac{\partial Q}{\partial X_1} \frac{X_2}{Q} \frac{\partial Q}{\partial X_2} = 1 \times \frac{a_{12} X_1 + a_{21} X_2}{Q}$$

which implies returns to scale are variable over the scale of production.

$E > 1$ if $a_{12}, \; a_{21} \geqslant O$

$E = 1$ if $a_{12}, \; a_{21} = O$ which means $Q = a_1 \; X_1 + a_2 \; X_2$

The ANH function is not constrained to be convex to the origin. The marginal rate of substitution is given by

$$\frac{dX_1}{dX_2} = - \frac{a_2 + a_{12} X_1 / X_2 + a_{21} X_1}{a_1 + a_{21} X_2 / X_1 + a_{12} X_2}$$

The elasticity of substitution is not constant and the function is a variable elasticity of substitution function. We now consider the Chu, Aigner and Frankel's (1970) log 16 quadratic law of production. Using L ($\geqslant$ 1) for labour, K ($\geqslant$ 1) for capital and $\bar{L}$, $\bar{K}$ for parameters which are, respectively, the maximising values of the labour and capital inputs that determine the highest total output, the Chu-Aigner-Franke 1 (CAF) function may be written

$$Q = A \left(\frac{L}{\bar{L}}\right)^{c_1 (1-\ln L/\ln \bar{L})} \left(\frac{K}{\bar{K}}\right)^{c_2 (1-\ln K/\ln \bar{K})}$$

or $\ln Q = a + a_1 \ln L + a_2 \ln K - b_1 (\ln L)^2 - b_2 (\ln K)^2$

where $a = \ln A - c_1 \ln \bar{L} - c_2 \ln \bar{K}$, $a_1 = 2c_1$, $a_2 = 2c_2$

$b_1 = c_1 / \ln \bar{L}$, $b_2 = c_2 / \ln \bar{K}$

The Chu-Aigner-frankel (CAF) function is nonhomogeneous and has nonconstant factor shares. It is obtained by simply adding the squared terms, $(\ln K)^2$ and $(\ln L)^2$ to the log linear Cobb Douglas function, and thus belongs to a family of logpolynomials. If we equate to zero, the marginal products

$$\partial Q/\partial L = 2c_1 (1-\ln L/\ln \bar{L})\, Q/L, \quad \partial Q/\partial K = 2c_2 (1-\ln K/\ln \bar{K})\, Q/K$$

we get $L = \bar{L}$ and $K = \bar{K}$. Since total output is maximised at this point, $\bar{L}$ and $\bar{K}$ may be called the maximum total productivity parameters. Similarly, since the average productivities Q/L and QK are maximised when $L = \bar{L}$ $1-1/2c_1$ and $K = \bar{K}$ $1-1$ $2c_2$, c_1 and c_2 are the maximum average productivity parameters. They determine the maximum average productivities once $\bar{L}$ and $\bar{K}$ are fixed. This helps determine the economic region of the production function.

For the CAF function the marginal product of labour exceeds the average product before the latter is maximum and is less than the average product after that, so the function obeys the law of variable proportions. This enables us to categorise the behaviour of input productivities and hence to determine the most economic region without attaching any significance to the symmetry of the stages of production.

The returns to scale are variable according to the values taken by L and K. Replacing L and K by λL and λK in the CAF function, we have

$$A (\lambda L/\bar{L})^{c_1 (1-\ln \lambda L/\ln \bar{L})} (\lambda K/\bar{K})^{c_2 (1-\ln/\ln \bar{K})}$$

$$= \left\{ (\bar{L}/L)^{c_1/\ln \bar{L}} \; (\bar{K}/K)^{c_2/\ln \bar{K}} \right\}^{\ln \lambda} Q$$

$$z^{\ln \lambda} \; Q, \text{ say}$$

If the inputs are increased by a multiple of λ, the output increased by multiple of $Z^{\ln \lambda}$ which itself is a function of inputs.

Janvry's (1972) generalised power production function (GPPF) allows for nonhomogeneity and also for variability of the returns to scale, marginal productivities, elasticities of production, marginal rates of substitution and elasticities of substitution. It includes as special cases the Cobb Douglas and transcendental production functions.

If $f_j(X)$ and $g(X)$ are polynomials of any degree in the arguments of the m dimensional input vector, X, the GPF may be written

$$Q = A \prod_{j=1}^{m} X_j^{f^j(X)} \; e^{g(X)}$$

This reduces to the Cobb Douglas form if $f_j(X) = \alpha_j$ for all j and $g(X) = O$. If $f_j(X) = \alpha_j$ for all j and $g(X) = \Sigma r_m X_m$, the transcendental form results.

The marginal product of factor X_j is

$$\frac{\delta Q}{\delta X_j} = Q\left(\frac{f^j(X)}{X_j} + \frac{\delta g(X)}{\delta X_j} + \sum_{j=1}^{m} \frac{\delta f^j(X)}{\delta X_j} \ln X \right)$$

which can assume positive, zero or negative values depending on the specification of the polynomials and hence can describe all three stages of production for $g(X) \neq O$.

The CPPF is homogeneous if and only if the polynomials $f_j(X)$, $j = 1, \ldots, m$, and $g(X)$

are homogenous of degree zero.

The function exhibits variable returns to scale unless all $f^m(X)$ are independent of X which reduces the GPPF to the Cobb Douglas form.

The economic region of production is defined by the set of values of the X's such that

$$0 < \sum_{j}^{m} f^{j}(X) \leq 1.$$

In the special two input case

$$Q = A X_1^{\alpha_1 + B_1 X_1} X_2^{\alpha_2} e^{r_1 X_1}$$

the marginal products are

$$\partial Q/\partial X_1 = (\alpha_1 + B_1 X_2 + r_1 X_1)\, Q/X_1$$

$$\partial Q/\partial X_2 = (\alpha_2 + B_1 X_2 \ln X_1)\, Q/X_2$$

For $X_1 = -(\alpha_1 + B_1 X_2)/r_1$, $\partial Q/\partial X_1 = 0$. It is maximum for

$\partial^2 Q/\partial X_1^2 = 0$ i.e. for $X_1 = -(\alpha_1 + B_1 X_2 + \sqrt{\alpha_1 + B_1 X_2})/r_1$

Thus X_1 has a positive and decreasing marginal product in the interval

$$-\frac{1}{r_1}(\alpha_1 + B_1 X_2 - \sqrt{\alpha_1 + B_1 X_2}) < X_1 < X_1 - \frac{1}{r_1}\alpha_1 + B_2 (X_2)$$

which is a function of X_2. Also X_1 has a negative marginal product if X_2 exceeds the critical level $-\frac{1}{B_1}(\alpha_1 + r_1 X_1)$

The elasticity of substitution of the GPPF is a variable parameter :

$$\sigma = \frac{b\,(a + b)}{b^2 = a\,\alpha_2 + 2b\,B_1\,X_2}$$

where $a = \alpha_1 + B_1 X_2$ and $b = \alpha_2 + B_1 X_2 \ln X_1$

If $B_1 = O$, $\sigma = 1$ which is the Cobb Douglas case.

If we introduce a suitable multiplicative exponential into Vinod's (1972) two input non-homogeneous production function, an extension of Janvry's form may be obtained. But the general form of Janvry's production function allows many more possibilities.

Kmenta Approximation

Kmenta approximation which was introduced as a Taylor series expansion upto the second order terms of the constant elasticity of substitution (CES) production function, is a commonly used relation in production function studies. We may look upon it as an obvious extension of Cobb Douglas function from which it may be obtained by the addition of some appropriate factors.

It is difficult to linearise and estimate the parameters of the CES function with nonconstant returns to scale

$$Q = r\left\{ \delta L^{-\rho} + (1-\delta)\, K^{-\rho} \right\}^{-v/\rho}$$

or $\ln Q/L = \ln r + (v-1) \ln L - \frac{v}{\rho} f(\rho)$

$$f(\rho) = \ln(\delta + (1-\delta)(K/L)^{-\rho})$$

$$= f(O) + \rho f^{1}(O) + \tfrac{1}{2} \rho^{2} f''(O) \text{ when expanded}$$

around $\rho = O$ with terms of order higher than the second omitted. Since

$$f(O) = O,\; f'(O) = -(1-\delta) \ln K/L$$

$$f''(O) = \delta(1-\delta).\,(\ln K/L)^2$$

We have

$$f(\rho) = -\rho\,\delta(1-\delta)\ \ln K/L + 1/2\rho^{2}\,\delta\,(1-\delta)\ \ln K/L^{2}$$

If we substitute $a_o = \ln r$, $a_1 = v-1$, $a_2 = v(1-\delta)$, $a_3 - 1/2\ v\,\rho\delta\ (1-\delta)$ we have

the Kmenta approximation $\ln Q/L = a_o + a_1 \ln L + a_2 \ln K/L + a_3 (\ln K/L)^{2}$ [3a]

or equivalently,

$$\ln Q = a_{10} + v\delta \ln K + v(1-\delta) \ln L - 1/2\,\rho\, v\,\delta\,(1-\delta)\,(\ln K - \ln L)^{2}$$

$$= a_{10} + a_{11} \ln K + a_{12} \ln L + a_{13} (\ln K - \ln L)^{2}$$

The last term on the right disappears if $\rho = 0$. The approximation is better with ρ closer to zero. If a_{13} is not significantly different from zero, the Cobb Douglas form may not be rejected though the exact situation would be unpredictable as a more general production function could result if a_{13} is significantly different from zero. Moreover a_{13} also depends on δ and $1=\delta$ and that makes the test weak.

The estimates of the parameters a_1 and a_2 in (3a) and hence of δ and ρ are not independent of the units of measurement. 1 So the elasticity of substitution may be evaluated at the mean level of a sample. The elasticity of substitution of this function depends on the input ratio and the function may be said to be homothetic. It may suffice to say that homotheticity implies that, if the expansion of the last term in the kmenta approximation with fresh coefficients viz., $a_3 (\ln K/L)^{2} = a_{31} (\ln K)^{2} - 2a_{32} \ln K . \ln L + a_{33} (\ln L)^{2}$

is tested in a linaer hypothesis framework, it results in $a_{31} = a_{32} = a_{32} = a_3$. If it dose not, a more general nonhomothetic polynomial function deserves to be considered. Specifically, Kmenta approximation belongs to the special class of homothetic productions in that its elasticity of substitution depends on the input ratio.

1 At K=L, The approximation is exact. For empirical work the units of K and L may be so chosen in the sample as to equate their geometric averages.

It is not necessary to expand f (ʕ) around ʕ = O. Any other appropriate value may be taken. But the error of approximation depends on the extent to which the actual value of ʕ deviates from the chosen value. It also depends on the input ratio as well as on the values of the other parameters in the function. The extent of the specification error resulting from the approximation depends on the closeness of the approximation.

Kmenta approximation of the CES function is linear in all parameters. Thus the best linear unbiased estimates can be obtained from it by using ordinary least squares method though the bias may have been caused by the dropping of the higher order terms.

The Translog Production Function

The production function underlying the cost theory is nonhomogeneous. The firm has increasing returns to scale at low output levels, constant returns to scale at intermediate levels and decreasing returns to scale at higher levels of output. Such a generalisation is not allowed by a homogeneous production function. A nonhomogeneous production may allow these variations.

We have seen that nonhomogeneity can manifest itself when terms of second and higher order are added to the Cobb Douglas function.

A nonhomothetic, generalised formulation of the Cobb Douglas and Kmenta functions may be written

$$\ln Q = \ln A + \alpha \ln k + B \ln L + r_{KK} (\ln K)^2 + r_{LL} (\ln L)^2 + r_{KL} \ln K \ln L)$$

whose scale elasticity is given by

$$E = \alpha + B + (2r_{KK} + r_{KL}) \ln K + (2) r_{LL} + r_{KL}) \ln L.$$

The bracketed terms of the scale elasticity vanish if $r_{KK} = r_{LL} = -\frac{r_{KL}}{2}$ in which case the function has constant scale elasticity and becomes homogeneous. It leads to the Cobb Douglas function if $r_{KL} = O$. The function can be useful in testing the homotheticity of the Kmenta approximation.

The expresion above is the two input case of the Christensen, Jorgensen and Lau's

(1971) translog production function. The translog production function may be considered as a second order local approximation of some underlying function. It has both linear and quadratic terms and can admit an arbitrary number of inputs. It may be viewed as an improved generalisation of the Cobb Douglas and Kmenta's CES approximation in that, with more than two inputs, and under reasonably general conditions it enables us to estimate partial elasticities of substitution among all forms of inputs. In the case of Cobb Douglas an CES functions, the separability conditions have to be imposed ie., specified a priori. In the translog case they can be tested.

Suppose there exists a technological relationship for output with three inputs ; capital (K), labour (L) and raw material (M), viz.,

$$\ln Q = \ln A + F(\ln K, \ln L, \ln M).$$

For the n input case $\ln Q = \ln A + F(\ln X_1, \ln X_2 \ldots, \ln X_n)$, a second order Taylor series approximation in the neighbourhood about the point with inputs unity results in

$$\ln Q - \ln A = F(0) + \sum_{i=1}^{n} \ln X_i \frac{\partial F}{\partial \ln X_i} + 1/2 \sum_{i}^{n} \sum_{j}^{n} \ln X_i \ln X_j \frac{\partial^2 F}{\partial \ln X_i \, \partial \ln X_j}$$

We have, in the three input case, with suitable notational changes,

$$F = \ln \alpha_0 + \alpha_K \ln K + \alpha_L \ln L + \alpha_M \ln M$$

$$+ 1/2\, r_{KK} (\ln K)^2 + 1/2\, r_{LL} (\ln L)^2 + 1/2\, r_{MM} (\ln M)^2$$

$$+ r_{KL} \ln K \ln L + r_{LM} \ln L \ln M + r_{MK} \ln M \ln K$$

By substituting $\ln A + \ln \alpha_0 = \ln A \alpha_0$, we have

$$\ln Q + \ln A \propto_o + \propto_K \ln K + \ldots \text{etc.}$$

For positive marginal products we must have

$$\partial \ln Q / \partial \ln X_i = \partial F / \partial \ln X_i = \propto_i + r_{iK} \ln K + r_{iL} \ln L + r_{iM} \ln M$$

$$> O \text{ for } i = K, L, M$$

For the function to be quasi concave at every data point, the bordered Hessian matrix should be negative and semidefinite.

It is found that the translog function, being a second order approximation and a quadratic, is not globally well behaved. But it may be considered as a good representation of production possibilities for most data.

Cobb Douglas Splines

Poirier's (1974) piecewise splines permit U shaped cost curves and piecewise homotheticity although differentiability of the functions along lines parallel to the input axes is no longer possible. In the Cobb Douglas function, returns to scale are nonvarying so that the average cost curve is not U shaped. With the Cobb Douglas splines the structural change and behaviour of the function in each piece can be tested.

Let $\propto i$ and Bj be positive constants and so chosen as to make the Cobb Douglas spline $F(K,L) = O_i K^{\propto_i} L^{B_j}$ continuous over the positive quadrants formed by the IJ rectangles defined by the knots in the meshes

$$L = \left[L_1 < L_2 < \ldots < L_J - I \right], \quad K = \left[K_1 < K_2 < \cdots < K_{I-1} \right]$$

Using the continuity conditions

$$\ln O_{i+i,j} = \ln O_{ij} + (\propto i - \propto i+1) \ln K_i \quad I = 1, 2 \cdots, I-1 \text{ for all } j$$

$$\ln O_{i,j+1} = \ln O_{ij} + (B_j - B_j + 1) \ln L_j \quad j = 1, 2 \cdots, J-1 \text{ for all } i,$$

and defining $\overline{K} = \max(\ln K - \ln K_i, O)$, $i = 1, 2, \ldots, I-1$

$$\bar{L} = \max(\ln L - \ln L_i, O) \quad j = 1, 2, \ldots, J-1,$$

We have for a given O and for all K and L

$$\ln F(K,L) = \ln O_{11} + L_1 \ln K + B_1 \ln L + \sum_{i=2}^{I-1} \propto_i K_i + \sum_{j=2}^{J-1} B_j L_j$$

where $\propto i$, Bj, represent changes in the output elasticities of K & L.

For a fixed output level Q_o, the isoquants over the rectangle [i,j]:

$$K = \left\{ Q_o L_j^{-B} / O_{ij} \right\}^{1/\propto i}$$ are continuous though having corners along the grid lines.

They are strictly convex if and only if each output elasticity is a decreasing step function of its respective output:

$$\propto_i \geqslant \propto_{i+1} \quad B_j \geqslant B_{j+1}, \quad i = 1, \ldots, I-1; \; j = 1, \ldots, J-1.$$

It can be shown that F (K,L) exhibits increasing returns to scale over all rectangles below and to the left of rectangle (i,j) and decreasing returns above and to the right of the rectable (i,j). It is indeed, possible to go for several other extensions and variations of the Cobb Douglas model. There is no doubt that many useful forms and modifications can be examined. But from a few forms of the production function models given above it can be seen that it is possible to arrive at a variety of extensions of the Cobb Douglas function by some simple contrivances, In particular the addition of new explanatory factors seems to help a lot. The present author carried out a number of empirical studies with several of these and other models. The industry data used were from several countries. In most cases it was realised that the addition of certain variables did contribute significantly to the explanatory power of the model concerned. At the same time it was found that the use of too many explanatory variables did not always help much. Several other interesting results were obtained with different models that were used. For instance, various comparable estimates of elasticity of substitution, returns to scale and other parameters were obtained. It is felt that complete reliance on a single form of production may sometimes, prove harmful. It the resources permit, it is advisable to try some alternatives with suitable explanatory variables depending on requirements.

Reference

Aigner, D. J. and C. 6. Chu (1968), *On estimating the industry production function*. American Economic Review. Vol. 58, p.626-39.

Allen R. G. D. (1967), *Mathematical analysis for economists*. MacMillan.

Arrow K. J., H. B. Chenery, *B. S. Minhas and R. M.* Solow (1961), Capital labour substitution and economic efficiency. Review of Economics and Statistics, Vol. 43, p. 225-50.

Berndt F, R. and L. R. Christensen (1973), *The translog function and the substitution of equipment, structures and labour in U. S. manufacturing*, 1929-68. Journal of Econometrics, Vol. 1, p. 81-113.

Brown M. and de Cani J. S. (1963), *Technological change and the distribution of income*. International Econ. Rev., Vol. 4, p. 289-309.

Chenery H. B. (1953), *Process and production functions from engineering data*: (*Studies in the structure of the American Economy dy Leontieff w. et al*), Oxford Univ. Press.

Chu S. F., D. J. Aigner and M. Frankel (1970), *On the log quardratic law of production*. Southern Econ. Journal, Vol. 37, p. 32-39.

Cobb C. W. and P. H. Douglas (1928), *A theory of production. American Econ.* Review, Supplement, Vol. 18, p. 139-65.

Halter A. N., H. O. Carter and J. G. Hocking (1957), *A note on the transcendental production function*. Journal of Farm Economics, Vol. 39, p. 966-74.

Heathfied D. G. (1971), *Production functions*. Mac Millan.

Hildebrand G. H. and T. C. Liu (1965), *Manufacturing production functions in the U. S. 1957*. Cornell Univ., Ithaca, N. Y.

Hilhorst J. G. M. (1962), *Measurement of productions in manufacturing industry*. Statistical studies No. 13. The Netherlands Central Bureau of Statistics. p. 7–29.

de Janvry A. (1972), *The generalised power production function*. Amer. Jour. of Agri; cultural Economics.

Kadiyala K. R. (1972) *Production functions and elasticity of substitution*. Southern Econn. Journal. Vol. 38. p.281–284.

Kmenta J. (1967), *On the estimation of the CES production function*. Inter. Econ. Rev. Vol. 8. p. 180–89.

Lavigne M. (1976), *Socialist economies, M. Robertson, London*. Layard P. R. G., J. D. Sargan, M. E. Ager and D. J. Jones (1971), Qualified manpower and economic perfo-rmance, The penguin press.

Monga G. S. (1979), *Some aspects of the neoclassical production function, Udyog pragati*, Vol. 3 No. 4, 17–21.

Monga G. S. (1980) *Some variants of the CES production Fucnction*, Vishleshan, Vol. 6, No. 2.73–99.

Poirier D. J. (1974), *On the use of Cobb Douglas splines. Working paper* 232. Univ. of Illinois.

Sudit E. (1973), *New types of nonhomogeneous production functions*. Ph. D. Disserta-tion, New York University.

Vinod H. (1972) *Non-homogeneous production functions and applications to telecom-munications*. Bell Journal of Economics and Management science. Vol. 3, p. 531–43.

Walters A. A. (1963), *Production and cost functions; an econometric survey*. Vol. 31, p. 1–66.