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A Generalised VES Production Function

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Often it is difficult to justify the use of a production function like the Cobb Douglas or the CES function which allows only nonvariable elasticity of substitution even though there may be reason to believe that in actual practice the case is otherwise. Homothetic production functions, which reflect variable returns to scale varying with output in a production process and which allow the testing of technical progress occurring in various forms of production functions, provide a partial solution to the problem. They allow the elasticity of substitution to be constant along a ray from the origin but not necessarily along an isoquant. Unfortunately, these production functions cannot be used at the microlevel because they do not obey the law of variable proportions so that the underlying microproduction functions remain unknown.

A less sophisticated form of production functions, the so called variable elasticity of substitution or VES functions, which concentrate mainly on the variability of elasticity of substitution, has been successfully used in some empirical studies. In this paper, we shall analyse some generalised forms of these VES functions.

Extensions of production function forms may be made by the use of the elasticity of substitution (δ) which is assumed to have a functional relationship with the input ratio or some other factor. This mode of derivation of production function forms is based on the argument that a constant or unitary elasticity of substitution is not a very realistic assumption.

The form of production function resulting from the relation between δ and the input ratio depends on the type of assumed relation. Moreover, there are two possibilities in the case

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of a two input function with K and L inputs. σ may be assumed to depend on L/K or K/L. In either case, we do away with the tacit assumption of Cobb Douglas and CES functions which require some kind of fixed technical substitution between inputs.

While it is difficult to predict the behaviour of elasticity of substitution, in practical cases it is certainly possible to derive some conclusions from observed results. Instead of assuming σ to be zero (Harrod Domar) or unity (Cobb Douglas) or infinity (straight ling isoquant) or even a constant between zero and infinity (CES function), it may be useful to assume that σ varies with the input ratio. Several possibilities may be considered.

σ is small at low input ratios, rising as the ratio rises, reaches a maximum and decreases at a certain value of input ratio.

Or, σ may be high at low input ratio, falling with a rise in the ratio.

Alternatively, σ is small at low input ratios and increases as the ratio rises. σ may be assumed to vary with any other factor or factors provided such a variation can be economically justified.

Unfortunately, it is not easy to incorporate various qualities expected of the parameter σ in the same function. Attempts have been made in the literature to deal with simpler cases.

Empirically, if σ is found not to vary significantly with capital deepening the validity of the CES function follows. But the constancy of elasticity of substitution is not a characteristic of the real world: it is counter intuitive. This has led to a search for suitable functions with variable elasticity of substitution. As early as 1931, Hicks, in his *Theory of Wages* emphasised that elasticity of substitution increased with an increase in capital and that should result in the making and adopting of a labour saving invention. A functional relation between elasticity of substitution and capital labour ratio was implied in Hicks' suggestions.

The concept of a variable elasticity of substitution (VES) production function, if confined to a particular function is not quite viable in as much as any production function which is not CES function is, by definition a VES function for which an infinity of possibilities can be discovered. Moreover, a VES function, in the sense in which it is being understood, just allows for one's unwillingness to assume constancy along an isoquant. In that case, one should be equally unwilling to assume constancy long a ray: this is the approach of the homothetic production functions. But what is needed is an algebraic form which is sufficiently general, linear in parameters and convenient to estimate. The VES functions, derived with the help of the elasticity of substitution relation, do not necessarily posses all these qualities. However, depending on the assumptions made about the elasticity of substitution, some of these qualities may be introduced into the functions that are derived.

We can arrive at some forms of production functions by giving different values to the elasticity of substitution. The VES functions will result by assuming that the elasticity of substitution is depedent on the capital labour ratio, K/L. ACMS (1961), in their study of the CES function, suggest such a dependence. Wise and Yeh (1965), in their inter-country study of wage and productivity differentials find that the elasticity of substitution increases to a certain point above unity as K/L increases and to less than unity as K/L decreases.

We will consider the forms developed by Sato (1965) and Revankar (1971.) Some more general forms will also be given. The following formulas will be found to be convenient in deriving these forms ($x = K/L$, $y = V/L$)

$$\sigma = -\frac{y'(y - xy')}{x\,y\,y''}$$

Here K = Capital, L = Labour, V = Value added. If it is assumed that $\sigma = 0$, we have $y/x = y'$. This leads to $y = Ax$ where A is a constant of integration. Since $x = K/L$, $y = V/L$, this relation may be written $V = AK$. Another possible solution is $V = BL$ if the formula for elasticity of substitution uses $1/x = L/K$ instead of $x = K/L$. If the equations $V = AK$ and $V = BL$ are both true but only one has meaning, we have the Leontief fixed proportions case:

$V = \min(AK, BL)$

If $\sigma = 1$, we have the differential equation:

$y'' + y'/x - y'^2/y = 0$

whose solution is $y = Ax^{\alpha}$ or $V/L = A(K/L)^{\alpha}$

which is the Cobb Dounglas function.

The assumption of σ = a constant and the resulting differential equation $y'' + y'/\sigma x - y'^2/\sigma y = 0$ easily lead to the CES function and its variants, Monga (1980). The explicit forms of some VES productions functions may be derived under the assumptions of perfect competition and constant technology. Let the elasticity of substitution be assumed to be a linear function of x or K/L. i.e.,

$$\delta(x) = \frac{y' \quad (y - x y')}{- x y y^{11}} = a + bx \quad \delta > O, x = \frac{K}{L}, y = \frac{V}{L}$$

substituting $u = \frac{y}{y,}$, we have

$$u - x = B \left(\frac{x}{a + bx} \right)^{\frac{1}{a}}$$

Here B > O because u – x = marginal rate of substitution > O. Writing A for an arbitrary constant

$$y = A \exp \int \frac{dx}{x + \left(\frac{B}{a + bx}\right)^{1/a}} \qquad A > O$$

If an explicit solution is desired for the above expression than a simplification may be introduced by assuming a to be a rational number equal to n/m where n, m are positive integers. By giving suitable values to a, Sato (1965) derives a variety of forms of production functions. Only some of these forms may be useful in practice.

If in the elasticity of substitution relation above, it is assumed that a = 1, then $\delta(x) = 1 + bx$ where x = K/L.

This allows a test of the null hypothesis b = O to fiind if the function should be a Cobb Douglas function. From this Revankar (1969) derives an explicit from of VES production functions given by

$$V = Z K^{1-\delta\rho} \left[L + (\rho - 1 K \right]^{\delta\rho}$$

where $b = \frac{\rho - 1}{1 - \delta\rho}$. The parameters ρ and ρ are affected by the units of measurement so that it is always possible to secure the condition $O < \rho < 1$ as a matter of convention.

To ensure that

$$\sigma = 1 + \frac{\rho\;1}{1-\rho\delta}\,\frac{K}{L} \geqslant 0$$

the restriction is $L/K \geqslant \frac{1-\rho}{1-\delta\rho}$

Revankar's function can be modified to be one with homgeneity of degree v:

$$V = ZK^{v\,(1-\delta\rho)}\left[L + (\rho-1)\;K\right]^{v\,\delta\rho}$$

The CES function cannot be derived from Revenkar's VES function. The latter is more general, however, in that as against a constant elasticity of substitution independent of the level of output, at all points of an isoquant, it has the substitution parameter constant only along a ray while varying along the isoquant.

If we write $L, = L + (\rho-1)\;K$
the function becomes

$$V = ZK^{v\;(1-\delta\rho)}\;L,^{v\,\rho\delta}$$

which means that the use of L instead of L_1 in the Cobb Douglas function involves specification error. With $\rho \neq 1$, L_1 may be regarded as a composite labour input in the contezt of the Cobb Douglas function.

Alternatively, writing $V = A\,(K/L)\,K^{v\;(1-\delta\rho)}\;L^{v\rho\delta}$

where $A\left(\frac{K}{L}\right) = \left[1+(\rho-1)\frac{K}{L}\right]^{v\delta\rho}$

we may compare it with $V = ZK^{v\;(1-\rho\delta)}\;L_1^{\;v\rho\delta}$

Revenkar's function satisfies the properties of a neoclassical production function. The factor shares are asymmetrical and nonconstant. They depend on the input ratio. Thus

$$S_K = \frac{K}{V}\frac{\delta V}{\delta K} = (1-\delta\rho) + \frac{v\delta\rho}{1+\frac{1}{\rho-1}\frac{L}{K}}$$

$$S_L = \frac{L}{V}\frac{\delta V}{\delta L} = \frac{v\delta\rho}{1+(\rho-1)\frac{K}{L}}$$

Revankar's VES function includes as special cases

Harrod Domar Case: for $\rho = 0$, $V = AK$

Cobb Douglas function: for $\rho = 1$, $V = AK^{v(1-\delta)} L^{\delta v}$

The St. Line Isoquant fn: for $\frac{1}{\rho} > 1$, $V = \frac{A}{\delta v}\left[\delta L + (1-\delta) K\right]^{v}$

Sato and Hoffman (1968) using the definition of elasticity of substitution derive a workable form of production function.

$$\text{Since } \sigma = \frac{y' (y-xy')}{-xyy''}$$

$$\therefore\ y = A \exp \int \frac{dx}{\frac{d\,\ln x}{Be^{x}} + x} = Ae^{\int h(x)\,dx} \text{ say}$$

To solve this it is necessary to get an explicity integration result for h(x). If S_L (x) stands for labour share, assume $h(x) = S_L(x)/x$.

Let S_L (x) be a lineer function of x :

S_L (x) = ax + x

then h(x) = a + 1

$$y = A e^{ax} x^b$$

Thus, by making suitable assumptions about an expression involving K/L for σ we nan arrive at a sufficiently general production function from which it is possible to generate a variety of forms of production functions. This form namely, $y = A e^{ax} x^b$ is very easy to handle and is usefull provided the assumptions associated with it are justified. The assumptions given above viz., $h(x) = S_L (x)/x$ and $S_L (x) = ax + x$ have been made only to arrive at a simple from for the production function. Sato and Hoffman fitted this VES function to U. S. and Dapatess time series data and found results which were more satisfactory than those obtained from the Cobb Douglas or CES function.

A simple extension of the above relation can be obtained by assuming the elasticity of substitution to be a quadratic function of capital ratio. Thus

$$\sigma (x) = a+b \left(\frac{K}{C}\right) + c \left(\frac{K}{C}\right)^2 = a + bx + cx^2$$

Substituting in the formula for σ we have the differential equation.

$$y' (x y' - y) = (a + bx + cx^2) x y y''$$

which leads to the production function

$$y = A \exp \int \frac{dx}{x + Bx^{1/a} (x-\alpha_1)^{\frac{c+b}{2ac}} (x-\alpha_2)^{\frac{c-b}{2ac}}}$$

where α_1 and α_2 are the roots of $a + bx + cx^2 = O$

Another extension canbe easily made by replacing the linear relation between σ and K/L by a more general relation.

$$\sigma (x) = a + bx^c$$

which when substituted in the formula leads to the production function

$$y = A \exp \int \frac{dx}{x + Bx^{1/a} (a + bx^{c})^{-1/ac}}$$

Explicit forms of production functions in these cases can be arrived at if some simplifying assumptions are made.

The dependence of σ on the capital labour ratio, K/L, has been justified on theoretical as well as empirical grounds. We have noted that Hicks suggested such a dependence as early as 1931 in his *Theory of Wages*. Although ACMS (1961) themselves did not make use of such a connection they did suggest it.

As noted earlier no definite rule about the behaviour of σ with K/L has been noticed On the same lines, we may expect the dependence of σ on the labour capital ratio L/K. As in the case of K/L ratio, the behaviour of σ with changes in L/K may not follow any definite rule. Since a high K/L implies a low L/K it is usual to think that a relation with one implies an inverse relation with the other. This may not be so in practice. It may be assumed that σ depends on both though not necessarily symmetrically and that the reaction of σ to changes in K/L need not obviate its reaction to changes in L/K. In other words, it is suggested that the capital intensity and labour intensity may not have predidcable inverse effects on elasticity of substitution. The substitution of capital for labour is a gradual process which has been going on for centuries. The substitution of labour for capital is relatively an uncommon phenomenon; it takes place sometimes under certain circumstances and in a manner, usually different from that of the substitution of capital for labour.

It may be useful to modify the assumption of σ depending on K/L alone. To avoid more complicated relations, let us assume a liner relation between σ and the two ratios K/L and L/K in the form

$$\sigma = a + b\,K/L + c\,L/K$$
$$= a + b\,x + c/x \qquad \text{where } x = K/L$$

where a, b, c, are unknown quantities. From this relation it is possible to arrive at an explicit form of production function if we use the definition of elasticity of substitution.

$$\sigma = - \frac{y' (y - xy')}{x \; yy''} = a + bx + \frac{c}{x}$$

which may be written $$(c + a2 + bx^2) \frac{y''}{y} - x \left(\frac{y'}{y}\right)^2 + \frac{y'}{y} = O$$

or $$(c + ax + bx^2) \left(1 - \frac{d}{dx} \frac{1}{u} \right) - x + \frac{1}{u} = O \text{ when } u = \frac{y'}{y}$$

If WE WRITE $g = - \int \frac{dx}{c + ax + bx^2}$, we have

$$\frac{d}{dx} \frac{e^g}{u} = e^g + x e^g \frac{d g}{d x}$$

$$\frac{y'}{y} = \left[x + b e^{-g (x)} \right]^{-1}$$

Hence $$y = A \exp \int \frac{dx}{x + Be^{-g (x)}}$$

where A, B are arbitrary constants. For an explicit solution of this we must have an explicit solution for g (x). We have

$$g(x) = - 2(4ab - c^2)^{-\frac{1}{2}} \tan^{-1} (2ax + c) (4ab - c^2)^{-\frac{1}{2}} \quad \text{if } 4ab > c^2$$

$$= 2(2ax + c)^{-1} \quad \text{if } 4ab = c^2$$

$$= -(c^2 - 4ab)^{-\frac{1}{2}} \ln \frac{2ax + c - (c^2 - 4ab)^{\frac{1}{2}}}{2ax + c + (c^2 - 4ab)^{\frac{1}{2}}} \quad \text{if } 4ab < c^2$$

Concentrating on the last case with $4ab < c^2$ and substituting

$$a_o = - \frac{1}{2a} \left[c - (c^2 - 4ab)^{1/2} \right]$$

$$b_o = \frac{1}{2a}\left[c + (c^2 - 4ab)^{1/2} \right]$$

$$c_o \quad (c^2 - 4ab)^{-1/2}$$

we get $$- g(x) = \ln \left\{ \frac{x + a_o}{x + b_o} \right\}^{c_o}$$

so that $$y = A \exp \int \frac{(x + b_o)^{c_o} \, dx}{x (x + b_o)^{c_o} + B (x + a_o)^{c_o}}$$

$$= A \exp \int \frac{P(x)}{R(x)} dx \text{ say}$$

where P (x), R (x) are polynominals.

The solution may be written

$$y = A (x - \alpha_1)^{\beta_1} (x - \alpha_2)^{\beta_2} \ldots\ldots (x - \alpha_n)^{\beta_n}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of R (x) and none of the roots is repeated so that

$$P(x)/R(x) = \sum_{i=1}^{n} \beta i (x - \alpha_i)^{-1}$$

If $\alpha_1, \alpha_2 \ldots\ldots, \alpha_n$ are repeated roots such that α_i is repeated m_i times $i = 1, 2, \ldots\ldots, q$

then $R(x) = (x \alpha_1)^{m_1} (x - \alpha_2)^{m_2} \ldots\ldots (x - \alpha_q)^{m_q}$ and

$$\frac{P(x)}{R(x)} = \sum_{i=1}^{q} \sum_{j=1}^{m i} \beta_{ij} (x - \alpha_i)^{-j} \quad \text{so that}$$

$$\int \frac{P(x)}{R(x)} dx = \sum_{i=1}^{q} \sum_{j=2}^{mi} \frac{\beta_{ij}}{i-j} (x - \alpha_i)^{-j+1} + \sum_{i=1}^{q} \beta_{ij} \ln (x - \alpha_i) + \ln A$$

where In A is a constant of integration. Writing $\beta_{ij} = \beta_i$ we have the explicit solution given by

$$y = A \left[\exp \sum_{i=1}^{q} \sum_{j=2}^{mi} \frac{\beta_{ij}}{1-j} (x - \alpha_i)^{1-j} \right] \prod_{i=1}^{n} (x - \alpha_i)^{\beta_i}$$

If we write $h_i(x) = (x - \alpha_i) P(x)/R(x)$ and its t^{th} derivative is denoted by $h_i^{(t)}$ we can find

$$\beta_{i\, m_i} = \frac{h_i^{(j-1)} (\alpha_i)}{(m_i - 1)!}$$

R(x) has repeated roots if R(x) and R′(x) have a common root. For instance, repeated roots are possible if $a_o = b_o$, i. e.

$$\frac{1}{2a} \left[c - (c^2 - 4ab)^{\frac{1}{2}} \right] = \frac{1}{2a} \left[c + (c^2 - 4ab)^{\frac{1}{2}} \right]$$

$$\text{or} \quad c^2 = 4ab$$

If each $m_i = 1$, the simple case of nonrepeated roots follows. Assuming nonrepeated roots and substituting $x = K/L$, we get the production relation

$$V = AL \left(\frac{R}{L} - \alpha_1 \right) \left(\frac{K}{L} - \alpha_2 \right) \cdots\cdots \left(\frac{K}{L} - \alpha_n \right)$$

The expression $V/L = A \prod_{i=1}^{n} (K/L - \alpha_i)$ is a polynomial in

K/L of degree $\sum_{i=1}^{n} \beta_i$

If all the roots are equal or if there is a single root the expression reduces to the form $V/L = A (K/L - \alpha)^{\beta}$ which is a pylynomial of degree β in K/L. This is the simplest expression that may be arrived at for practical work unless $\alpha = O$ in which case it is reduced to the Cobb Douglas form with constant returns to scale. In this form α may be considered as a correction factor for K/L.

The production relation

$$\frac{V}{L} = A \left(\frac{K}{L} - \alpha \right)^{\beta}$$

satisfies neoclassical requirements if $O < \frac{\beta}{x - \alpha} < 1$ so that

$$\frac{\delta V}{\delta L} = \left(1 - \frac{\beta x}{x - \alpha} \right) \frac{V}{L} > O$$

$$\frac{\delta V}{\delta K} = \frac{\beta}{x - \alpha} \frac{V}{K} > O$$

the MRS is given by $s = \frac{1-\beta}{\beta} x - \frac{\alpha}{\beta}$ so that the elasticity of substitution is a function of 1/x in this simple case:

$$\sigma = 1 - \frac{\alpha}{1-\beta} \frac{1}{x}$$

which implies $a = 1,\ b = O,\ c = -\frac{\alpha}{1-\beta}$

and which reduces to $\delta = 1$ if $\alpha = O$

The estimation of this relation requires the use of non-linear regression technique unless the value α is known. It should be possible to estimate the relation more easily if α (or $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ in the n-root case) can be found. This may be done by giving appropriate values to a, b, c. If c=o we will have the Sato case. If c = o, a = 1, it will be the Revankar case. From the results obtained above it is obvious that a study along these lines can lead us to more realistic forms of production functions which have not received ample attention mainly because of the associated complicated expressions and also because this line of attack has remained neglected while some extensions in other directions have been popular. The main difficulty in these forms is that of the values of a, b, c. If these are known it may become quite easy to arrive at the explicit forms in most cases-In all these extensions described above it may be possible to take a = 1 or a = O and begin with the assumption $\delta(x) = 1 + bx$ and test the null hypothesis b = O to find if the production function is a Cobb Douglas function. If b is found to be significant its value can be used. In any of the other possible extensions viz. $\delta = a + bx + cx^2$, $\delta = a + bx^c$, $\delta = a + bx + \frac{c}{x}$ etc. and a suitable value of c, determined. It is not necessary to begin with a = 1. It is as well if any other convenient value of a is taken. It may also be possible, if circumstances permit, to assume suitable values of b or hoth a and b. In any case some experimentation for suitable values of a, b, c can be fruitful to arrive at explicit forms of production functions and to proceed for empirical work.

The production function developed here gives an interesting possibility of extension of a known result. The use of the form evolved may require a lot of work but some simplifications may help without distorting the useful elements in it. In that case this form would have a better theoretical justification as well as empirical attractiveness. The availability of the computer should make the statistical estimation easy.

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