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About Riesz theory of compact operators

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Abstract

In this paper, it is shown that every compact operators are bounded and continuous. The bounded and continuous properties of an operator is sufficient for a Riesz operator. For mapping T: K-λI in normed linear space with some extended [1] properties, T becomes compact.

Keywords: Chain length; Riesz operators; Relatively regular operator

1. Introduction

For the mapping T = K-λI, in normed linear space, [where I is the identity map, K is compact map and $\lambda \neq 0$ is an arbitrary scalar from ϕ], the fredholm alternative theorem is valid. The null spaces and image spaces of T^k, where k is natural number, form chains with the same chain length. The image space B(T) is always closed; K-λI is open and the eigenvalues of K form a null sequence, provided infinitely many such eigenvalues exist. An operator which satisfies these properties for all $\lambda \neq 0$, is known as a **Riesz operator**. If K is a compact map, then T = K-λI possesses a decomposition into an invertible operator R(λ) and an operator S(λ) of finite rank. This leads to the decisive definitions. The operators that in the quotient algebra of all continuous maps with respect to the ideal of all finite dimensional maps, they are right or left invertible respectively, such invertible are known as Atkinson operators. Still another important notion which stands together very closely with the above definitions. If for a continuous operator T, there exists a continuous operator S with T.S.T. = T, then T is called **relatively regular operator [2]**, For a topological characterization of these relatively regular operators, if E is a named linear space and $K \in R(E)$; then put I-K=T and N(T^k)=N_k, B(T^k)=B_k for k=0,1,2…

2. Some properties of null and image spaces

As null spaces form an increasing sequence $\{0\}=N_0 \subset N_1 \subset N_2 \subset N_3 \subset \ldots$.

1. **N_k is closed:** This follows from the continuity of T^K, k = 0,1,2….
2. **dim $N_1 < \infty$:** If $x_n \in N$, and $\|x_n\|=1$, $n \in N$, then (I-K)x_n=0, therefore x_n=K. x_n; on account of the compactness of K, the sequence {k.x_n} has a subsequence {k.x_n} with

$$\lim_{k\to\infty} K.\, x_n = \lim_{k\to\infty} x_{n_k} = y$$

Therefore $\{x \in E : \| x \| = 1\} \cap N_1$ is compact, and hence N_1 is finite dimensional.

3. **dim $N_k < \infty$:** Since

$$T^k = (I-K)^k = I - \left[k.K - \frac{k(k-1)}{2} K^2 + ... + (-1)^{K-1}.K^k \right]$$

$= I - K_k,\ k \in N$, holds and K_k is compact [3]. From above result it follows that dim $N_k < \infty$ for all k = 0,1,2…

4. If there exists $k \geq 0$ s.t. $N_k = N_{k+1}$, then it follows that $N_k = N_{k+1},\ n \in N$.

<u>Proof:</u> If $x \in N_{k+2}$, then $T^{k+2}.x = T^{k+1}(T.x) = 0$, $T.x \in N_{K+1} = N_k$, Therefore, also $T^k(T.x) = T^{k+1}.x = 0$ i.e., $x \in N_{k+1}$ Hence $N_{k+1} = N_{k+2}$ and by complete induction, wet get $N_k = N_{k+n}$.

<u>Definition (1:1):</u> There exists, either smallest number $p = p(T) < \infty$ satisfying $N_p = N_{p+1}$ in which case we say that T fulfills the null chain condition with the null chain length p - or we always have $N_k \neq N_{k+1}$, therefore $p(T) = \infty$ in which case we say that T does not fulfill the null chain condition [4].

<u>Theorem (2.1):</u> If T = I-K, $K \in R(E)$, then N_k is closed, dim $N_k < \infty$ for all k = 0,1,2…, and T satisfies the null chain condition. We now verify the corresponding statements for image spaces:

B_k is closed: In case k = 1, we must show that $y \in \overline{B_1}$, $y \in B_1$, in other words

$y_n = T.x_n \in B_1$ and $y_n \to y$, then we must show that $y \in B_1$, if

(1) $\delta_k = \underset{u \in N_1}{\text{glb}} \| x_k - u \|$, then we choose an $u_k \in N$, s.t. $\|x_k - u_k\| \leq 2.\delta_k$: we put $v_k = x_k - u_k$, then

(2) $\|v_k\| \leq 2.\delta_k$: and $y_k = T.x_k = T(v_k + u_k) = T.v_k$ we first of all show that the sequence $\{u_k\}$is bounded [5]; otherwise let there exists a subsequence which we again denote by $\{v_k\}$ s.t. $\|v_k\| \to \infty$ for $k \to \infty$. For $w_k = \frac{v_k}{\| v_k \|}$. The sequence

(3) $\{T.w_k\} = \left\{ T.\frac{v_k}{\| v_k \|} \right\} = \left\{ \frac{y_k}{\| y_k \|} \right\}$ converges to 0. Since $\{y_k\}$ converges by assumption. Since $\{w_k\}$ is a boundend s.t, $\{K.w_k\}$ contains a convergent subsequence $\{K.w_{k_j}\}$. Then by (3) it follows that W_{k_j} = (I-K) W_{k_j} + k. $W_{k_j} \to z$. And also by (3) lim T. W_{k_j} = T.z = 0, hence $z \in N_1$ from which it follows that.

$$\|w_k - z\| = \left\| \frac{x_k - u_k}{v_k} - z \right\|$$

$$= \frac{1}{\| v_k \|}.\left\| x_k - (u_k + \| v_k \|.z) \right\|$$

$$\geq \frac{\delta_K}{\| v_k \|}$$

$$\geq \frac{1}{2}$$

Since $u_k + \|v_k\|.z \in N_1$. This contradicts that lim $w_{k_j} = z$: hence $\{v_k\}$ is bounded. Therefore the sequence {K. v_k }contains a convergent subsequence {K. v_{k_i} } and the sequence{ v_{k_i} }= {T. v_{k_i} +k. v_{k_i} }={ $y_{k_i} + K.u_{k_i}$ } converges to an element v. It follows that $y_{k_i} = v_{k_i} - K v_{k_i} \to v - K.v = T.v$, Hence $y = T.v \in B_1$ [6]. Then from above equation (3) it follows that B_k is closed also for k>1.

Theorem (2.2)

a) If $T \in L(X)$ fulfills the null chain condition with the null chain length $p \le m$ iff $N_r \cap B_m = \{0\}$ for some $r \in N$ holds or: $N_m = N_{r+m}$ iff $N_r \cap B_m = \{0\}$, $r \in N$

b) $T \in L(X)$ fulfills the image chain condition with the image chain length $q \le n$ iff there exists a component C_r to B_r in X which is contained in N_n.

or : $B_n = B_{n+r}$ iff $X = C_r \oplus B_r \subset N_n + B_r = X$, $r \in N$ [7].

Theorem (2.3): If $K \in R(E)$, then $\alpha(I-K) = \beta(I-K) < \infty$, K possesses a representation K = R+S with RS= SR = 0, $R \in R(E)$ and $S \in R(E)$; I – K possesses a representation (I-R)-S with (I-R)S = S(I-R), where I-R possesses a continuous inverse on E.

Definition (1.2): $L \in \phi$ is called eigenvalue of $S \in L(X)$, X an arbitrary vector space, if there exists an $x \in X$, $x \neq 0$ s.t. (S - L.I).x = 0, therefore $\alpha(S - L.I) > 0$, x is then called eigensolution corresponding to L; N(S-L.I) is called the eigenspace corresponding to L and $\alpha(S - L.I)$ the multiplicity of L.

Theorem (2.4): If $K \in R(E)$, then for every $\lambda \neq 0$, then

a) $\alpha(K - \lambda I) = \beta(K - \lambda I)$
b) $p(K - \lambda I) = q(K - \lambda I)$
c) $B(K - \lambda I)$ is closed
d) $K - \lambda I$ is open
e) the eigenvalues of K form, provided they are infinite in number, a null sequence.

Theorem (2.5): If If $K \in L(E)$, then every $\lambda \neq 0$, then

a) $\alpha(K - \lambda I) = \beta(K - \lambda I)$
b) $p(K - \lambda I) = q(K - \lambda I)$
c) $K - \lambda I$ is relatively regular.
d) the eigenvalues of K form a null sequence, provided they are infinite in number.

3. Conclusions

$K \in L(E)$ is called a Riesz- operators if $K - \lambda I$ possesses the properties (a) – (e) of theorem (2.4) for each $\lambda \in \phi$. Again $K \in L(E)$ is a Riesz operator iff for every $\lambda \neq 0$, it has the properties (a) – (d) of theorem (2.5). Thefore, every compact operator is always a Riesz-operator.

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