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Almost boundedness and matrix transformation

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Abstract

The sequence space a_c^r have been defined and the various classes of infinite matrices have been characterized by Aydin and Başar, (see, [1]) , where $1\leq p \leq \infty$. In this paper we characterize the classes $(a_c^r : f_\infty)$, $(a_c^r : f)$ and $(a_c^r : f_0)$,where f_∞ , f and f_0 denote respectively the spaces of almost bounded sequences , almost convergent sequences and almost convergent null sequences.

Keywords: Sequence space of non-absolute type;, almost convergent sequences; β-duals and Matrix Transformations.

1. Introduction, Background and Preliminaries

A sequence space is defined to be a linear space with real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively.

Let ω denote the space of all sequences (real or complex). Let X and Ybe two non-empty subsets of ω. Let A =(a_{nk}) ,$(n, k \in \mathbb{N})$, be an infinite matrix of real or complex numbers. We write $(Ax)_n = A_n(x) = \sum_k a_{nk}x_k$. Then $Ax = \{A_n(x)\}$ is called the A-transform of x, whenever $A_n(x) = \sum_k a_{nk}x_k < \infty$ for all $n \in \mathbb{N}$. We write $\lim_n Ax = \lim_n A_n(x)$. If $x \in X$ implies $Ax \in Y$, we say that A-defines a matrix transformations from X into Y, denoted by $A: X \to Y$. By $(X: Y)$, we mean the class of all matrices A such that $A: X \to Y$.

For a sequence space, the matrix domain $X_A$of an infinite matrix A is defined as

(1) $X_A = \{x = (x_k) \in \omega: Ax \in X\}$

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences $x = \{x(n)\}_{n=0}^{\infty}$ with supremum norm $\|x\| = sup_n|x(n)|$. Let T denote the shift operator on ω, that is , $Tx = \{x(n)\}_{n=1}^{\infty}$, $T^2x = \{x(n)\}_{n=2}^{\infty}$ and so on. A Banach limit $L$is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator and $L(e) = 1$, where $e = (1,1,\dots)$ (see, [2]), that is, a functional $L: \ell_\infty \to \mathbb{R}$ is called a Banach limit if

(i) L is linear,

(ii) $L(x) \geq 0$ if $x_n \geq 0$ for all n.

(iii) $L(x) = L(Tx)$ where T is shift operator on ω.

(iv) $L(e) = 1$, where $e = (1, 1, \dots)$.

Since the Hahn-Banach norm preserving extension is not unique, there must be many Banach limits in the dual space of ℓ_∞, and usually different Banach limits have different values at the same element in ℓ_∞. However, there indeed exists sequences whose values of all Banach limits are same. If $x = \{x_n\}_{n=0}^{\infty} \in c$, where c is a Banach space of ℓ_∞ consisting of convergent sequences, then $L(x) = \lim_n x_n$ is a trivial example. Besides this there also exists non-convergent sequences satisfying this property. For example $x = \{1, 0, 1, 0, \dots\}$ the value of $L(x) = \frac{1}{2}$ is same for every Banach limit. Lorentz (see, [4]) called a sequence $x = \{x_n\}_{n=1}^{\infty}$ almost convergent if all Banach limits of x, $L(x)$, are same, and this unique Banach limit is called F-lim of x. In his paper Lorentz proved the following criterion for almost convergent sequences.
A $x = \{x_n\}_{n=0}^{\infty} \in \ell_\infty$ is almost convergent with F-limit $L(x)$ if and only if

$$\lim_{p\to\infty} t_{mn}(x) = L(x),$$

where, $t_{mn}(x) = \frac{1}{p}\sum_{i=0}^{p-1} T^i x_n$, $(T^0 = 0)$, uniformly in $n \geq 0$.

The above limit can be rewritten in detail as

$$(\forall\varepsilon > 0), (\exists p_0)(\forall p > p_0)(\forall n)\left|\frac{x_n + \cdots + x_{n+p-1}}{p} - L\right| < \varepsilon.$$

We denote the set of almost convergent sequences by f.

$$f = \{x \in l_\infty : \lim_m t_{mn}(x)\ exists,\ uniformly\ in\ n\}.$$

Nanda [6] has defined a new set of sequences f_∞ as follows:

$$f_\infty = \{x \in l_\infty : \lim_m |t_{mn}(x)| < \infty\}.$$

We call f_∞ the set of all almost bounded sequences. The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., (see, [1, 5, 7]).
Following (see, [1], [7]), the sequence space a_c^r is defined as the set of all sequences whose A^r-transform is in c, that is,

$$a_c^r = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n+1}\sum_{k=0}^{n}(1 + r^k)x_k \ exists \right\}$$

;

where , $a_{nk}^r = \begin{cases} \dfrac{1+r^k}{n+1} \ , & 0 \le k \le n, \\ 0 \ , & k > n. \end{cases}$

With the notation of (1) that, $a_c^r = (c)_{A^r}$.

2. Main Results

Define the sequence $y=(y_k(r))$ which will be used, by the A^r-transform of a sequence $x=(x_k)$, that is,

(2) $$y_k(r) = \sum_{j=0}^{k} \frac{1+r^j}{k+1} x_j \ ; \ \text{for } k \in \mathbb{N}.$$

For brevity in notation, we write

$$t_{mn}(Ax) = \frac{1}{m+1}\sum_{j=0}^{m} A_{n+i}(x) = \sum_k a(n,k,m)\, x_k$$

where , $a(n,k,m) = \dfrac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k}$; $(n,k,m \in \mathbb{N})$

Also , $$\tilde{a}(n,k,m) = \Delta\left[\frac{a(n,k,m)}{1+r^k}\right](k+1) = \left[\frac{a(n,k,m)}{1+r^k} - \frac{a(n,k+1,m)}{1+r^{k+1}}\right](k+1)$$

We denote by X^{β}, the $\beta-$deal of a sequence space X and mean the set of all the sequences $x=(x_k)$ such that $xy=(x_k y_k) \in cs$ for all $y=(y_k) \in X$. Now, we give the following lemmas which will be needed in proving the main Theorems.

Lemma 2.1[1]: Define the sets $D_1(p)$ and $D_2(p)$ as follows

$$D_1^r = \left\{ a=(a_k) \in \omega : \sum_k \left| \Delta\left(\frac{a_k}{1+r^k}\right)(k+1) \right| < \infty \right\}$$

$$D_2^r = \left\{ a=(a_k) \in \omega : \left(\frac{a_k}{1+r^k}\right) \in cs \right\}$$

where , $$\Delta\left(\frac{a_k}{1+r^k}\right) = \frac{a_k}{1+r^k} - \frac{a_k}{1+r^{k+1}}$$

Then , $$\left[a_c^r\right]^{\beta} = D_1^r \bigcap D_2^r$$

Lemma 2.2 [5]: $f \subset f_{\infty}$

Theorem 2.1: $A \in (a_c^r : f_{\infty})$ if and only if

(3) $$\sup_{n,m\in\mathbb{N}}\sum_k |\tilde{a}(n,k,m)| < \infty$$

and

(4) $$\left\{\frac{a_{nk}}{1+r^k}\right\}_{k\in\mathbb{N}} \in cs \text{, for all } n\in\mathbb{N}$$

Proof: Sufficiency: Suppose the conditions (3) & (4) holds and $x\in a_c^r$. Then $\{a_{n,k}\}_{k\in\mathbb{N}} \in [a_c^r]^\beta$ for every $n\in\mathbb{N}$,the A-transform of x exists. Since $x\in a_c^r$, by hypothesis, and $a_c^r \cong c$ (see,[1]) , we have $y\in c$. Thus , we can find $K>0$ such that $\sup_k |y_k| < K$.

$$|t_{mn}(Ax)| = \left|\sum_k a(n,k,m)x_k\right| = \left|\sum_k \tilde{a}(n,k,m)y_k\right|$$

$$\leq \sum_k |\tilde{a}(n,k,m)||y_k| \leq K\sum_k |\tilde{a}(n,k,m)|$$

Taking $\sup_{m,n}$ on both sides, we get $Ax\in f_\infty$ for every $x\in a_c^r$.

Necessity: Suppose that $A\in(a_c^r : f_\infty)$. Then Ax exists for every $x\in a_c^r$ and this implies that $\{a_{n,k}\}_{k\in\mathbb{N}} \in [a_c^r]^\beta$ for every $n\in\mathbb{N}$, the necessity of (4) is immediate. Now, $\sum_k a(n,k,m)\,x_k$ exists for each m , n and $x\in a_c^r$, the sequences $\{a(n,k,m)\}_{k\in\mathbb{N}}$ define the continuous linear functionals $\psi_{mn}(x)$ on a_c^r by

$$\psi_{mn}(x) = \sum_k a(n,k,m)\,x_k \quad ; (n,k,m\in\mathbb{N}).$$

Since a_c^r and c are norm isomorphic (see [1] ,), it should follow with (2) that $\|\psi_{mn}(x)\| = \|\tilde{a}(n,k,m)\|$ holds for every $k\in\mathbb{N}$. This implies that the functionals defined by ψ_{mn} on a_c^r are point wise bounded, so by uniform bounded principle , there exists $M>0$ such that

$\|\psi_{mn}(x)\| \leq M$ for every $m,n\in\mathbb{N}$.

Thus we conclude that

$$\sup_{m,n}|\psi_{mn}(x)| = \sup_{m,n}\left|\sum_k a(n,k,m)x_k\right| = \sup_{m,n}\left|\sum_k \tilde{a}(n,k,m)y_k\right| < M$$

This implies that $\sup_{n,m\in\mathbb{N}}\sum_k |\tilde{a}(n,k,m)| < \infty$, which shows the necessity of the condition (3) and the proof of (i) is complete. □

Theorem 2.2 : $A\in(a_c^r : f)$ if and only if (3) ,(4) and

(5) $$\lim_m \tilde{a}(n,k,m) = \beta_k \text{ ,uniformly in } n\text{, and for each } k\in\mathbb{N}.$$

(6) $$\lim_m \sum |\tilde{a}(n,k,m) - \beta_k| = 0\text{, uniformly in } n.$$

Proof: Sufficiency: Suppose that the conditions (3), (4), (5) and (6) hold and $x \in a_c^r$. Then Ax exists and at this stage , we observe with the help of (5) & (6) that

$$\sum_{j=0}^{k} \left| \beta_j \right| = \sup_{m,n} \sum_j \left| \tilde{a}(n, j, m) \right| < \infty$$

holds for every k .This gives that $(\beta_k) \in l_1$ .Since $x \in a_c^r$ by hypothesis and $a_c^r \cong c$ (see,[1]) , we have $y \in c$ .Therefore , we can easily see that $(\beta_k y_k) \in l_1$ for each $y \in c$ and also there exists $K > 0$ such that $\sup_k |y_k| < K$. Now for $\varepsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$,there is some $m_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=0}^{k_0} \langle \tilde{a}(n,k,m) - \beta_k \rangle y_k \right| < \frac{\varepsilon}{2}$$

for every $m \geq m_0$ and $k_0 \in \mathbb{N}$.

Also by (6) , there is some $m_1 \in \mathbb{N}$, such that

$$\sum_{k=k_0+1}^{\infty} \left| \tilde{a}(n,k,m) - \beta_k \right| < \frac{\varepsilon}{2}$$

for every $m \geq m_1$ uniformly in n .Thus , we have

$$\begin{aligned} \left| \frac{1}{m+1} \sum_{j=0}^{m} (Ax)_{n+j} - \sum_k \beta_k y_k \right| &= \left| \sum_k \langle \tilde{a}(n,k,m) - \beta_k \rangle y_k \right| \\ &\leq \left| \sum_{k=0}^{k_0} \langle \tilde{a}(n,k,m) - \beta_k \rangle y_k \right| + \sum_{k=k_0+1}^{\infty} \left| \langle \tilde{a}(n,k,m) - \beta_k \rangle y_k \right| \\ &< \frac{\varepsilon}{2} + \sum_{k=k_0+1}^{\infty} \left| \langle \tilde{a}(n,k,m) - \beta_k \rangle \right| \left| y_k \right| \\ &< \frac{\varepsilon}{2} + K \frac{\varepsilon}{2K} = \varepsilon \end{aligned}$$

for all sufficiently large m ,uniformly in n . Hence, $Ax \in f$, which proves sufficiency.

Necessity: Suppose that $A \in (a_c^r : f)$. Then , since $f \subset f_\infty$ (by Lemma 2.2), the necessities of (3) and (4) are immediately obtained from Theorem 2.1 . To prove the necessity of (5), consider the sequence $b^{(k)}(r) = \left(b_n^{(k)}(r) \right)$ for every $k \in \mathbb{N}$, where

$$b_n^{(k)}(r) = \begin{cases} (-1)^{n-k} \dfrac{1+k}{1+r^k} & , \quad k \leq n \leq k+1 \\ 0 & , \quad 0 \leq n < k \text{ or } n > k+1 \end{cases}$$

Since Ax exists and is in f for each $x \in a_c^r$, one can easily see that

$Ab^{(k)}(r) = \left\{ \Delta\left(\frac{a_{nk}}{1+r^k} \right)(k+1) \right\}_{n\in\mathbb{N}} \in f$ for all $k \in \mathbb{N}$,which proves the necessity of (6). Similarly taking $x = e \in a_c^r$, we shall get

$$Ax = \left\{ \sum_k \Delta\left(\frac{a_{nk}}{1+r^k} \right)(k+1) \right\}_{n\in\mathbb{N}} \in f$$

,which proves the necessity of (5).This concludes the proof.

Note that if we replace f by f_0 , then Theorem 2.2 is reduced to the following corollary:

Corollary: $A \in (a_c^r : f_0)$ if and only if (3) ,(4), (5) and (6) holds with $\beta_k = 0$ for each $k \in \mathbb{N}$.

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