BIBECHANA
Vol. 22, No. 1, April 2025, 41-51

ISSN 2091-0762 (Print), 2382-5340 (Online)
Journal homepage: http://nepjol.info/index.php/BIBECHANA
Publisher:Dept. of Phys., Mahendra Morang A. M. Campus (Tribhuvan University)Biratnagar

# Dynamics of wter pollution concentration with uniform and exponential increment of pollutants

**Jeevan Kafle[1], Ranjan Kumar Chaudhary[1], Bekha Ratna Dangol[2,*]**

[1]*Centarl Department of Mathematics, Tribhuvan University, Kathmandu, Nepal*
[2]*Department of Mathematics,Patan Multiple Campus, Tribhuvan University Lalitpur, Nepal*
[*]Corresponding author. Email: bdangol37@gmail.com

**Abstract**

*The study focuses on the one dimensional Advection-Dispersion Equation (ADE) to study the dynamics of concentration of water pollution when the pollutants at the source is increasing uniformly and exponentially. Analytical solutions are obtained by using Laplace transform and numerical solutions are by Finite Difference Method (FDM). The steady state case is studied. The dynamics of pollution concentration along the length of river channel are shown through two dimensional plots by varying the rate of added pollutants, cross sectional area of river and water flow velocity. The pollution concentration decreases along the length of river (downstream) for each analysis. The analytical and numerical solutions are shown in three dimensional plots. The analytical and numerical solutions are compared with the help of relative error. The relative errors calculated for uniform and exponential increment of pollutants at the source are compared while studying the dynamics of the concentration. The environmental and chemical engineering may substantially benefit from such studies.*

**Keywords**
Advection, Dispersion, Pollutant Concentration, Laplace transform, Finite Difference, Relative error.

**Article information**
Manuscript received: May 13, 2024; Revised: January 14, 2025; Accepted: January 16, 2025
DOI https://doi.org/10.3126/bibechana.v22i1.65799
This work is licensed under the Creative Commons CC BY-NC License. https://creativecommons.org/licenses/by-nc/4.0/

## 1 Introduction

The rapid and unplanned development of urbanization, unregulated industries and highly dense population in the urban areas are becoming a serious problem in the natural environment [1]. More specifically, the degradation of the natural environment are due to poor waste management, public transports, excessive use of pesticides in agriculture, deforestation, release of toxic materials from industries, unmanaged drainage and many more [1]. In some Eastern philosophies, water is often metaphorically linked to human civilization. However, contamination in these water resources like pond, river and lake at the urban areas poses a significant challenge to the communities, vegetation and other species that rely on them [1]. The major causes for these water pollution are unmanaged growth of urbanization, industries, and lack of public awareness. Indeed, rivers play a crucial role in aquatic ecosystems by transporting water and nutrients to various areas [2]. However, when contaminated with pollutants, these waterways can become hazardous to human health and other species. Consumption of contaminated water or exposure to pollutants can lead to sickness and even death among both humans as well as wildlife. These highlight the critical importance of protecting water sources from pollution [3]. As per a recent World Health Organization (WHO) report, inadequate water quality is responsible for 3.1% of fatalities and contributes to 80% of illnesses [4]. In the study of water pollution, biochemical oxygen demand (BOD) serves as the gauge for assessing water pollution levels [5]. This analysis employed most of the parameters outlined by Pimpunchat et al. (2007), under the assumption that the predominant pollutants are biochemical wastes [6]. For instance, Bagmati river is widely recognized as the most polluted river in Nepal with its biochemical oxygen demand (BOD) consistently on the rise [7] (See, Fig. 1). The extreme BOD is achieved due to the consequence of high levels of organic matter, high iron, and low dissolved oxygen [8]. Industrial waste and drainage from urban dwellers are responsible for the main source of BOD loads, leading to the increased concentration of the pollutants along the river. Prior studies on the water quality of Bagmati river revealed the degradation in its lower portion with several significant parameters that exceeded the limits set by the National Surface Water Quality Standards and Classification [7]. The ongoing deterioration of water quality in the Bagmati river, revered as a holy river, serves as a compelling motivation for researchers to conduct systematic studies on water pollution. Pollution in river channel takes place due to disposal of waste materials from point and non-point pollution sources [5].

Understanding the transport and dispersion of pollutants from the source to body across the medium can substantially contribute for identifying sources of pollution, assessing the impact of contaminants on aquatic ecosystems, guiding and implementing pollution control measures and facilitating public awareness [1]. In overall, understanding, managing, and mitigating the impacts of pollution on water resources are the important aspects in the study of water pollution [6]. To describe the pollution dispersion in the water body, physical and mathematical models are developed. Physical models are small scale representations of the water body. However, numerical experiments using mathematical models can cover the larger domain. These mathematical models are divided into statistical and deterministic models [1]. Statistical models rely on the analysis of historical monitoring data, while deterministic models are constructed based on mathematical description of physical processes.

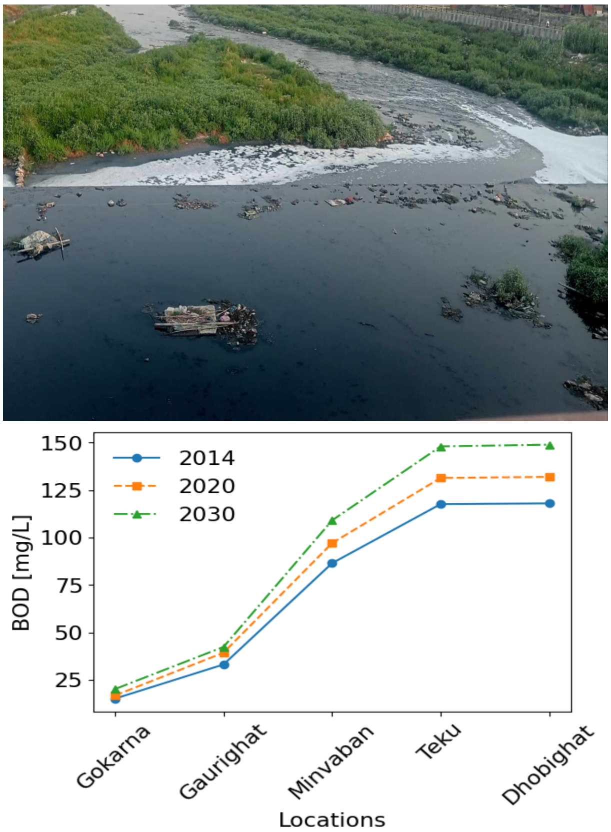



Figure 1: Polluted Bagmati river (Left) and BOD Status in Bagmati river at major locations (cities) at different years (Right) [8]. BOD is in rise over time (t) and location (x), where the river flows downstream. More specifically, BOD contents consistently exceeded the limit for moderately contaminated water (2-8 mg $L^{-1}$) and even crossed the threshold for treated municipal sewage (20 mg $L^{-1}$).

For the mathematical models to study water pollution, various conservation equations have been developed to describe the movement of substances subject to both advection and diffusion [9]. Advection refers as the transport of substances by the flow of water and dispersion refers to the spreading of substances due to concentration gradients, typically from areas of higher concentration to areas of lower concentration [6]. With the rise of environmental engineering and the need to model the transport of pollutants in underground and surface water, the advection-dispersion equation gained prominence and is developed during the latter half of the 20th century [9]. Due to the reliability, the advection and dispersion equations have been emerged in several fields, including environmental sciences, chemistry, and physics. These equations are incredibly employed for the spreading of pollution which may take the form of any substance (gas, liquid or solid) or energy ( radioactivity, heat, sound, or light) [10].

Marusic [11] developed a model to describe the dispersion of pollutants in systems resembling rivers over the temporal changes in pollutant concentrations. Pochai et al. [12] proposed a convection-diffusion model with constant coefficients to optimize contaminant levels in waste water treatment and successfully solved it by employing the finite element method. The primary objective was to reduce the initial cost of water treatment, thereby achieving more cost-effective treatment solutions. Later on, it was extended to two dimensions by Tabuenca et al. [13]. The advection-diffusion equations are employed for various prospective: prediction of the transport concentration of pollutants (Johari et al. [14]); contamination of rivers (Pimpunchat et al. [6]); deriving pollutant concentration field (Miller et al. [15]). Poudel et al. [?, 10], solving one dimensional ADE for pollutant concentration with zero dispersion and steady state cases are analyzed for oxygen concentration with zero as well as non zero dispersion coefficients. Finding the analytical and numerical solutions for the developed models are another important aspects of this field. Lots of efforts have been made in finding analytical solutions in some states and reduced situations using different approaches. The example includes the use of Laplace transform and with Greens functions for pollutant concentration (Carslaw and Jaeger [17]; Poudel et al. [10]); finite difference approach in semi-infinite mediums (Savovic and Djordjevich [18]); semi infinite domain with zero initial concentration (Van and Alves [9]); Laplace transform approach for the temporally and spatially dependent solute dispersion in a one-dimensional semi-infinite porous medium (Kumar et al. [19]).

The purpose of the study is to obtain the analytical and numerical solutions of advection dispersion equation modeled for uniform and exponential increment of pollution concentration at point source and hence observe the dynamics of pollution concentration at different spatial points along the length of the river. Laplace transform technique is used for analytical solution and Finite Difference Scheme, Forward Time Central Space Scheme (FTCSS) is used for numerical solution. The obtained solutions are compared with the help of relative error for more reliable result.

## 2 Mathematical Model and Numerical Method

### 2.1 The Governing Equation

In this model developed by Manitcharoen and Pimpunchat (2020) [5], the involved parameters are based on the assumption that the contaminants are only due to the biochemical wastes [6]. The rate of change of the pollution concentration with position $(x)$ and time $(t)$ is expressed as

$$\frac{\partial(AP)}{\partial t} = D_p\frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(vAP)}{\partial x} - K_1\frac{X}{X+k}(AP) + qH(x), \quad 0 \le x < L,\ t > 0 \qquad (1)$$

where, $H(x)$ is the Heaviside function as suggested by Chapra [20] (1997), defined as

$$H(x) = \begin{cases} 1, & 0 < x < L, \\ 0, & otherwise. \end{cases} \qquad (2)$$

Here, $v$ is the flow velocity of water along the river, $P$ is the pollution concentration, $D_p$ is the dispersion coefficient, $K_1$ is the pollutant degradation rate coefficient, $q$ is added pollutant rate, $k$ is half-saturated oxygen demand concentration, $X$ represents concentration of dissolved oxygen, and $A$ is the cross sectional area of the river [5]. Pollutant of oxygen concentration quantities are considered homogeneous throughout the cross-section of river and can fluctuate over the its length. This presumption meets the requirement as proposed by Dobbin [21]. For convenience, we assume that the parameters $A$, $q$, $D_p$, and $K_1$ remain constant both spatially and temporally [22]. The pollution concentration at the source may increase uniformly or sometimes, it may increase exponentially. The case of insignificant $k$ ($k \approx 0$) is taken into consideration for analysis as it isnot possible to use Laplace transform technique to solve advection dispersion equation [5]. From equation (1), the rate of change of the pollution concentration $P_1(x,t)$ resulting from a uniform increase in the pollution concentration at source is

$$\frac{\partial(AP_1)}{\partial t} = D_p\frac{\partial^2(AP_1)}{\partial x^2} - \frac{\partial(vAP_1)}{\partial x} - K_1AP_1 + qH(x). \qquad (3)$$

Similarly, the pollution concentration $P_2(x,t)$ resulting from exponential rise in the pollution concentration at source is

$$\frac{\partial(AP_2)}{\partial t} = D_p\frac{\partial^2(AP_2)}{\partial x^2} - \frac{\partial(vAP_2)}{\partial x} - K_1AP_2 + q(1-\exp(-\lambda x))H(x), \qquad (4)$$

where the arbitrary constant $\lambda$ represents the exponential pollution source term. Initially, the domain is free of solutes. So, the initial condition is

$$P(x,t) = 0, \quad x \ge 0,\ t = 0. \qquad (5)$$

The concentration at the origin is $P_0$ and the concentration gradient is supposed to be zero at infinite length of course. So, boundary conditions are

$$P(x,t) = P_0, \quad x = 0,\, t > 0, \tag{6}$$

$$\frac{\partial P(x,t)}{\partial x} = 0, \quad x \to \infty,\, t > 0. \tag{7}$$

## 2.2 Analytical Solution

Let the function $P(x,t)$ is assumed to be bounded and defined for $t > 0$. Let $\bar{P}(x,s)$ be Laplace transform of the function $P(x,t)$, where $s$ is transformed variable of $t$ [5]. Using Laplace transform for equations (3) and (4), we get respectively,

$$D_pA\frac{d^2\bar{P}_1(x,s)}{dx^2} - vA\frac{d\bar{P}_1(x,s)}{dx} - K_1A\bar{P}1(x,s) + A(s\bar{P}_1(x,s) - P_1(x,0)) + \frac{q}{s} = 0, \tag{8}$$

$$D_pA\frac{d^2\bar{P}_2(x,s)}{dx^2} - vA\frac{d\bar{P}_2(x,s)}{dx} - K_1A\bar{P}_2(x,s) + A(s\bar{P}_2(x,s) - P_2(x,0)) + \frac{q(1-\exp(-\lambda x))}{s} = 0, \tag{9}$$

where $s > 0$, $x \geq 0$.
The initial (5) and boundary conditions (6) and (7) are transformed respectively as

$$\bar{P}(x,s) = 0, \quad x \geq 0,\, s = 0; \quad \bar{P}(x,s) = \frac{P_0}{s}, \quad x = 0,\, s > 0; \quad \frac{d\bar{P}}{dx}(x,s) = 0, \quad x \to \infty,\, s > 0. \tag{10}$$

Using (10) in (8), we get

$$\frac{d^2}{dx^2}(\bar{P}_1(x,s)) - \frac{v}{D_p}\frac{d}{dx}(\bar{P}_1(x,s)) - \frac{K_1+s}{D_p}\bar{P}_1(x,s) = -\frac{q}{sAD_p}. \tag{11}$$

Auxiliary equation of (11) is

$$m^2 - \frac{v}{D_p}m - \frac{K_1+s}{D_p} = 0. \tag{12}$$

This is quadratic in $m$ whose roots are

$$m_1 = \frac{v + \sqrt{v^2 + 4D_p(K_1+s)}}{2D_p}, \quad \text{and} \quad m_2 = \frac{v - \sqrt{v^2 + 4D_p(K_1+s)}}{2D_p}.$$

Thus, complimentary function and particular integral respectively are,

$$c_1e^{m_1x} + c_2e^{m_2x} \quad \text{and} \quad \frac{q}{As(K_1+s)}.$$

The solution of (8) is

$$\bar{P}_1(x,s) = c_1e^{m_1x} + c_2e^{m_2x} + \frac{q}{As(K_1+s)}. \tag{13}$$

Using boundary condition (10),

$$c_1 + c_2 = \frac{P_0}{s} - \frac{q}{sA(K_1+s)}. \tag{14}$$

Differentiating equation (13) with respect to $x$, we get

$$\frac{d\bar{P}_1(x,s)}{dx} = c_1m_1e^{m_1x} + c_2m_2e^{m_2x}.$$

Again, by using boundary condition (10) in above equation, we get $c_1 = 0$. Thus from (13), required solution is

$$\bar{P}_1(x,s) = \left(\frac{P_0}{s} - \frac{q}{sA(K_1+s)}\right)e^{m_2x} + \frac{q}{As(K_1+s)}.$$

Also, we can write

$$\bar{P}_1(x,s) = \left(\frac{P_0}{s} - \frac{q}{sA(K_1+s)}\right) \exp\left(\left(\gamma - \sqrt{\frac{s+\beta^2}{D_p}}\right)x\right) + \frac{q}{As(K_1+s)}, \tag{15}$$

where, $\gamma = \frac{v}{2D_p}$, $\beta = \sqrt{\frac{v^2}{4D_p} + K_1}$.
Now, we find the solution of equation (9) by the similar process as above. Using (10) in (9), we get

$$\frac{d^2}{dx^2}(\bar{P}_2(x,s)) - \frac{v}{D_p}\frac{d}{dx}(\bar{P}_2(x,s)) - \frac{K_1+s}{D_p}\bar{P}_2(x,s) = -\frac{q(1-\exp(-\lambda x))}{sAD_p}. \tag{16}$$

Auxiliary equation of (16) is

$$m^2 - \frac{v}{D_p}m - \frac{K_1+s}{D_p} = 0. \tag{17}$$

So, roots of equation (17) are

$$m_1 = \frac{v + \sqrt{v^2 + 4D_p(K_1+s)}}{2D_p}, \quad \text{and} \quad m_2 = \frac{v - \sqrt{v^2 + 4D_p(K_1+s)}}{2D_p}.$$

Thus complimentary function and particular integral respectively are,

$$c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{and} \quad \frac{q}{As(K_1+s)} - \frac{qe^{-\lambda x}}{As(K_3+s)} \quad \text{where,} \quad K_3 = K_1 - \lambda v - \lambda^2 D_p.$$

Thus, the solution of (16) is

$$\bar{P}_2(x,s) = \left(\frac{P_0}{s} - \frac{q}{sA(K_1+s)} + \frac{qe^{-\lambda x}}{As(K_3+s)}\right) e^{m_2 x} + \frac{q}{As(K_1+s)} - \frac{qe^{-\lambda x}}{As(K_3+s)}. \tag{18}$$

Using boundary conditions (10) and calculating same as above, we get the solution of equation (16) as

$$\bar{P}_2(x,s) = \left(\frac{P_0}{s} - \frac{q}{sA(K_1+s)} + \frac{qe^{-\lambda x}}{As(K_3+s)}\right) e^{m_2 x} + \frac{q}{As(K_1+s)} - \frac{qe^{-\lambda x}}{As(K_3+s)}.$$

It can be written as

$$\begin{aligned}\bar{P}_2(x,s) = &\left(\frac{P_0}{s} - \frac{q}{sA(K_1+s)} + \frac{qe^{-\lambda x}}{As(K_3+s)}\right) \exp\left(\left(\gamma - \sqrt{\frac{s+\beta^2}{D_p}}\right)x\right) \\ &+ \frac{q}{As(K_1+s)} - \frac{qe^{-\lambda x}}{As(K_3+s)}.\end{aligned} \tag{19}$$

Taking inverse Laplace transform to the equations (15) and (19), the respective analytical solutions of equations (3) and (4) are

$$\begin{aligned}P_1(x,t) = &\frac{q}{AK_1}(1-\exp(-K_1 t)) + \frac{1}{2}\left(P_0 - \frac{q}{AK_1}\right)\exp\left(\left(\frac{\beta}{\sqrt{D_p}} + \gamma\right)x\right)\text{erfc}\left(\frac{x}{2\sqrt{D_p t}} + \beta\sqrt{t}\right) \\ &+ \exp\left(\left(\frac{-\beta}{\sqrt{D_p}} + \gamma\right)x\right)\text{erfc}\left(\frac{x}{2\sqrt{D_p t}} - \beta\sqrt{t}\right) + \frac{q}{2AK_1}\exp\left(\frac{v}{D_p}x - K_1 t\right) \\ &.\text{erfc}\left(\frac{x}{2\sqrt{D_p t}} + \gamma\sqrt{D_p t}\right) + \exp(-K_1 t)\text{erfc}\left(\frac{x}{2\sqrt{D_p t}} - \gamma\sqrt{D_p t}\right),\end{aligned} \tag{20}$$

$$P_2(x,t) = \frac{q}{AK_1}(1-\exp(-K_1t)) - \frac{q}{AK_3}\exp(-\lambda x)(1-\exp(-K_3t)) + \frac{1}{2}\left(P_0 - \frac{q}{AK_1} + \frac{q}{AK_3}\right)$$
$$.\exp\left(\left(\frac{\beta}{\sqrt{D_p}}+\gamma\right)x\right)\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}+\beta\sqrt{t}\right)+\exp\left(\left(\frac{-\beta}{\sqrt{D_p}}+\gamma\right)x\right)$$
$$.\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}-\beta\sqrt{t}\right)+\frac{q}{2AK_1}\exp\left(\frac{v}{D_p}x-K_1t\right)\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}+\gamma\sqrt{D_pt}\right)$$
$$+\exp(-K_1t)\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}-\gamma\sqrt{D_pt}\right)-\frac{q}{2AK_3}\exp\left(\left(\frac{\theta}{\sqrt{D_p}}+\gamma\right)x-K_3t\right)$$
$$.\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}+\theta\sqrt{t}\right)+\exp\left(\left(\frac{-\theta}{\sqrt{D_p}}+\gamma\right)x-K_3t\right)\text{erfc}\left(\frac{x}{2\sqrt{D_pt}}-\theta\sqrt{t}\right), \tag{21}$$

where, $\theta = \sqrt{v^2/4D_p + v\lambda + D_p\lambda^2}$.

## 2.3 The steady state case

Taking $t\to\infty$ to the equations (20) and (21) gives the steady state case as follows

$$P_1(x,t\to\infty) = \frac{q}{AK_1} + \left(P_0 - \frac{q}{AK_1}\right)\exp\left(\left(\gamma - \frac{\beta}{\sqrt{D_P}}\right)x\right), \tag{22}$$

$$\text{and}\quad P_2(x,t\to\infty) = \frac{q}{AK_1} - \frac{q}{AK_3}\exp(-\lambda x) + \left(P_0 - \frac{q}{AK_1} + \frac{q}{AK_3}\right)\exp\left(\left(\gamma - \frac{\beta}{\sqrt{D_p}}\right)x\right). \tag{23}$$

Now, taking $x\to\infty$ to calculate downstream pollutant concentration, we get

$$P_1(x\to\infty,t\to\infty) = P_2(x\to\infty,t\to\infty) = \frac{q}{AK_1}. \tag{24}$$

## 2.4 Numerical technique

Here, we use Forward Time and Central Space Scheme (FTCSS) to find numerical solution of equations (3) and (4). These equations in finite difference form can be written as [5],

$$\frac{P_{1m}^{n+1}-P_{1m}^{n}}{k} = \frac{D_p}{h^2}(P_{1m+1} - 2P_{1m}^n + P_{1m-1}^n) - \frac{v}{2h}(P_{1m+1}^n - P_{1m-1}^n) - K_1P_{1m}^n + \frac{q}{A}, \tag{25}$$

$$\frac{P_{2m}^{n+1}-P_{2m}^{n}}{k} = \frac{D_p}{h^2}(P_{2m+1} - 2P_{2m}^n + P_{2m-1}^n) - \frac{v}{2h}(P_{2m+1}^n - P_{2m-1}^n) - K_1P_{2m}^n + \frac{q}{A}(1-\exp(-\lambda x_m^n)) \tag{26}$$

where, $m$ and $n$ refer to the discrete step size $h$ and $k$ respectively. The initial and boundary conditions (10) can be written respectively as [5]

$$P_{m,0}=0,\quad x\geq 0;\quad P_{0,n}=P_0,\quad t>0;\quad P_{M,n}=P_{M-1,n},\quad x\to\infty,\, t\geq 0. \tag{27}$$

The stability condition is $D_pk/h^2 \leq 1/2$ for the above schemes.

## 3 Results and Discussion

We comparatively study the dynamics of the water pollution concentration over the river channel when the pollutants at the source increases uniformly and exponentially along with the variation of the rate of added pollutant, cross section area of the water channel, and flow velocity over time. The parametric values are extracted from the different existing literatures. For various plots to study the concentration dynamics, the values for involved parameters are taken as: $D_p$=10 $\text{m}^2\,\text{hr}^{-1}$,

$v$=4 m hr$^{-1}$, $A$=10 m$^2$, $K_1$=1 hr$^{-1}$, $P_0$=1 kg m$^{-3}$, $q$=1.02 kg m$^{-1}$ hr$^{-1}$ and $\lambda$=0.06289 [5].

### 3.1 Comparison of dynamics of concentrations $P_1$ and $P_2$

Keeping the same parameter values, the dynamics of the pollution concentrations $P_1$ along the river for the uniformly increasing pollutants and $P_2$ for exponentially increasing pollutants are revealed in the Fig. 2.

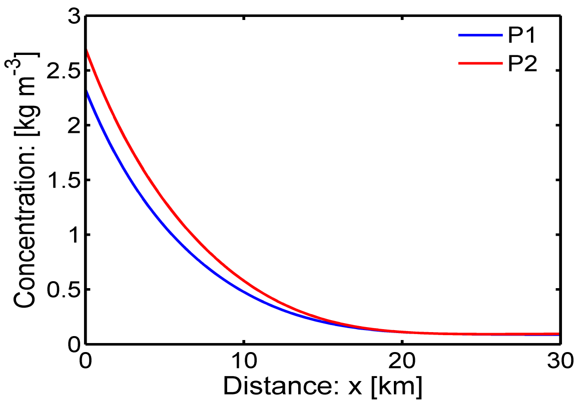



Figure 2: Comparison of pollution concentrations $P_1$ and $P_2$ at different spatial regions when the pollutant at source increases uniformly and exponentially respectively. The concentration $P_2$ is higher than $P_1$ at the vicinity of the source of pollutant at $x = 0$ and then both saturates to the almost same level as the water flows substantially downstream.

Initially, at the source $x = 0$, pollution concentration $P_2$ is relatively larger than that of $P_1$ as pollutants at the source is increasing exponentially for $P_2$ and uniformly for $P_1$. Both the concentrations decrease non-linearly over the distance and time until the flow travels 18 km. In this moment, the rate of decrement of $P_2$ is substantially faster than that of $P_1$. After traveling about 18 km from the source, both the concentrations decreases asymptotically. As a result, both graphs look like the same. However, they vary by the incredibly smaller margin. The applied model enables to study the dynamics for the significantly longer travel distance. However, we study only upto 30 km of travel distance. The concentration decreases rapidly for exponential increment case rather than uniformly increment form along the length of the river channel.

### 3.2 Dynamics of concentrations $P_1$ and $P_2$ as rate of added pollutant varies

The concentration dynamics of pollution $P_1$ and $P_2$ are shown in Fig. 3 as the rate of added pollutants $q$ varies for $q = 0.04, 1.02$ and 2.00. The dispersion rate of the pollution is the greatest when the rate of added pollutants $q$ is the least. On the contrary, the rate is the least when $q$ is the greatest.

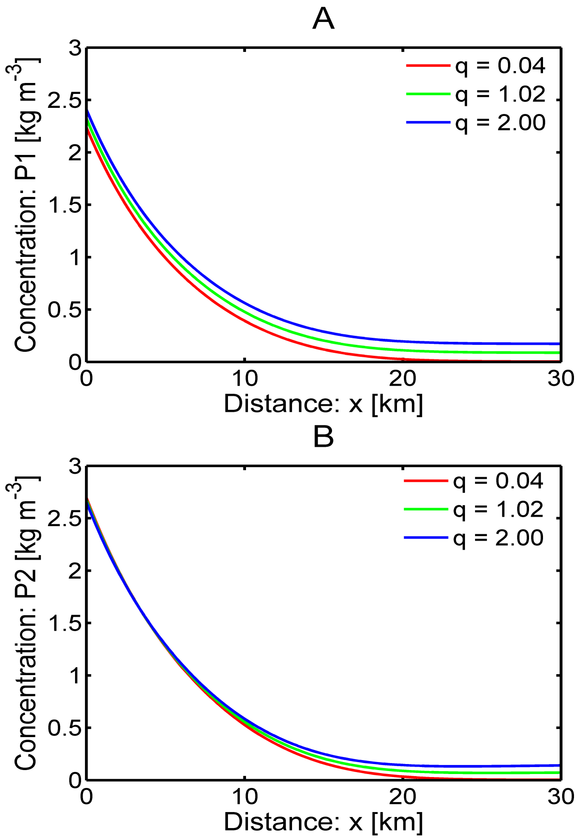



Figure 3: Evolution of pollution concentration **A**: $P_1$, **B**: $P_2$ at different spatial regions along the river as rate of added pollutants ($q$) varies. The pollution concentration decreases at slow rate as the rate of added pollutants increases.

As a result, the effect of rate of added pollutants $q$ near the upstream is very less and that near the downstream is dominant in both the cases. But in case of Fig. 3B, the effect of rate of added pollutants $q$ is very less near the origin as it is affected by exponential source term $\lambda$. For the case Fig. 3A, the concentration decreases non-linearly till $x$=19 km and then after it goes asymptotically whereas for the case Fig. 3B, it decreases non-linearly till $x$=21 km and then after goes asymptotically. The concentration of pollutants decreases at slow rate at any cross section of the river as the rate of added pollutants $q$ increases.

### 3.3 Dynamics of concentration $P_1$ and $P_2$ as cross section area of the river channel varies

The variation of pollution concentrations $P_1$ and $P_2$ with different cross sectional area $A$ of the river channel, along the length of the river is shown in Fig. 4 as the cross sectional area of the river varies for A=5, 10, 80.

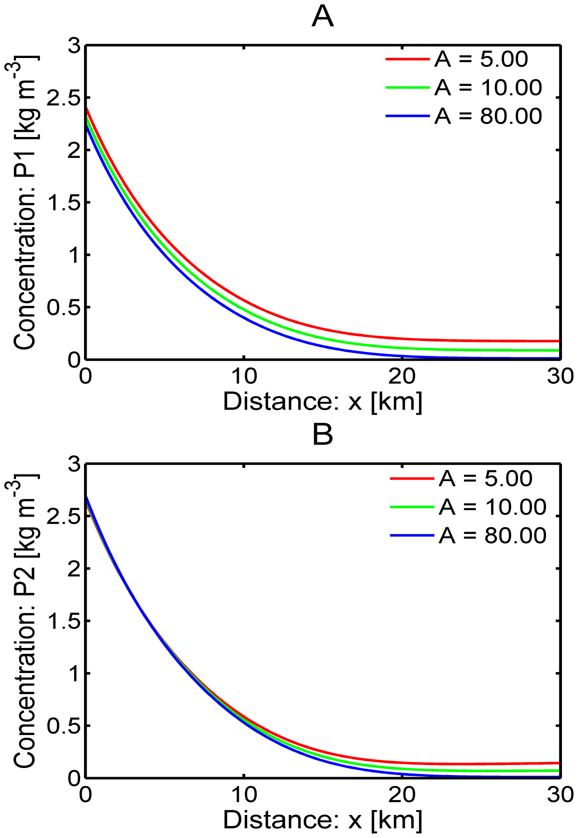



Figure 4: Evolution of pollution concentration **A**: $P_1$, **B**: $P_2$ at different spatial regions along the river as cross sectional area A of the river channel varies. The pollution concentration decreases as the cross sectional area of the river channel increases.

When the cross sectional area of the river channel is less, then there is not enough space for pollutants to be dispersed more rapidly. Thus, the concentration of pollutants remains condensed and is more for least area. Thus, the pollution concentration decreases at any cross section of the river as the cross sectional area increases. The rate of decrement of concentration is more when the cross sectional area is more and is less for less area. The effect of cross sectional area is less for case Fig. 4A and very less for case Fig. 4B near the upstream and dominant near the downstream. The increment of pollutants is exponential in case of Fig. 4B, so the concentration profile near the origin seems to be very close. But it is not so close in case of uniform increment. For the case Fig. 4A, the concentration decreases non-linearly till $x$=21 km and then after it goes asymptotically whereas for the case Fig. 4B, it decreases non-linearly till $x$=23 km and then after goes asymptotically. In overall, the concentration decreases along the length of river channel as cross sectional area increases.

### 3.4 Dynamics of concentration $P_1$ and $P_2$ as flow velocity of the river varies

The variation of pollution concentrations $P_1$ and $P_2$ with different water flow velocity $v$ along

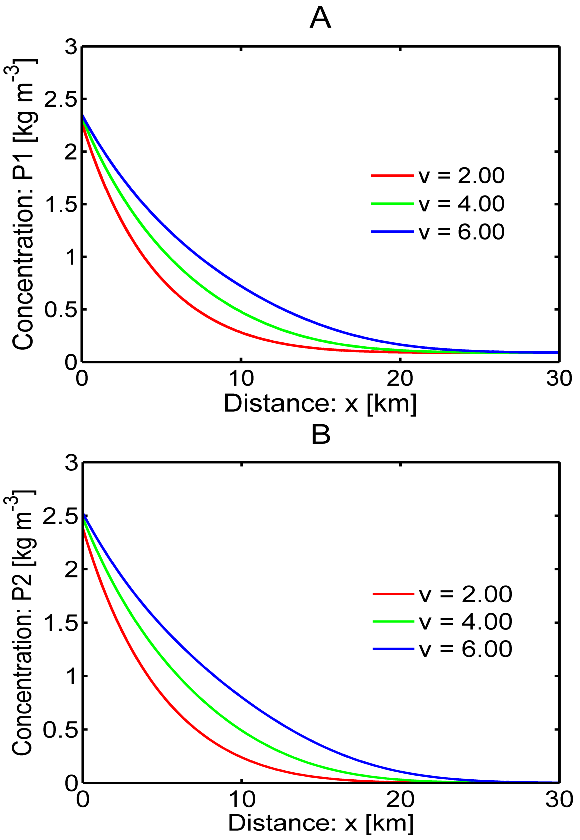



Figure 5: Evolution of pollution concentration **A**: $P_1$, **B**: $P_2$ at different spatial regions along the river as the flow velocity of water $v$ varies. The pollution concentration increases as the flow velocity of water increases.

the length of the river is shown in Fig. 5. With the increase of velocity, advection becomes more dominant than dispersion and hence the rate of decrement of concentration of the pollution decreases with increase in water flow velocity along the length of the river. If the water flow velocity is least, then pollutants get dispersed very near to the origin and concentration decreases so rapidly. But when water flow velocity is high then the concentration does not get chance to disperse so rapidly at the same position and hence rate of decrement of concentration becomes slower. For $v$=2, 4 & 6, concentration decreases non-linearly till the position $x$=16, 20 & 23 km in case of Fig. 5A, and $x$=17, 21 & 25 km in case of Fig. 5B, respectively and then goes asymptotically.

### 3.5 Dynamics of the concentration as travel distance and travel time both varies

We obtained the analytical and numerical solutions of Advection Dispersion Equation (ADE) with uniform and exponential increment of pollutants at origin by employing Laplace transform technique and Finite difference technique respectively.

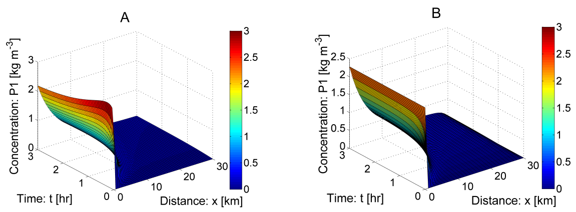



Figure 6: Analytical: **A** and numerical: **B** solution of equation (3). The both solutions do not provide exactly same values but provide nearly comparable values.

Fig. 6 shows the pollution concentration $P_1(x,t)$ at each grid point of time and space. In both cases Fig. 6A and Fig. 6B, concentration at origin is the highest and goes down as distance increases. The concentration decreases rapidly along the distance whereas it decreases slowly along time. The concentration at peak point i.e at $x$=30, $t$=3, is very less approximate to zero but not actually zero. Though values of concentration obtained from analytical and numerical solutions are not exactly same, they are close to each other. Thus nature of both the three dimensional plots are same but not exact.
Similarly, Fig. 7 shows the nearly same thing as described for Fig. 6 i.e. concentration at origin is the highest and decreases rapidly along the distance and slowly along the time. But concentration at origin for Fig. 7 is higher than that of Fig. 6. This clears that Forward Time Central Space Scheme (FTCSS) is somehow good scheme to solve these modeled equations but not perfect.

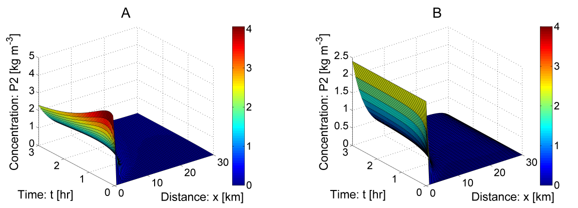



Figure 7: Analytical: **A** and numerical: **B** solution of equation (4). The both solutions do not provide exactly same values but provide nearly comparable values

### 3.6 Error Analysis

The solutions obtained analytically and numerically can be compared with relative error which is defined by [5],

$$\text{Relative error} = \left| \frac{P_{\text{analytical}} - P_{\text{numerical}}}{P_{\text{analytical}}} \right| \tag{28}$$

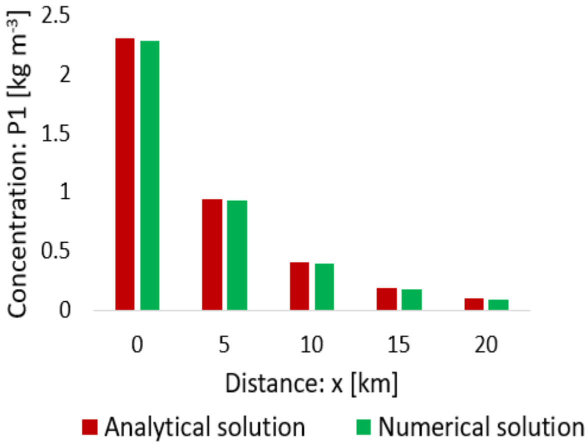
The relative errors of pollution concentrations for both the cases by $h$=1 and $k$=0.03 at $t$=3 hours are tabulated below. The domain for distance is taken up to 20 km from the origin to analyze the relative error between analytical and numerical solution of Advection-Dispersion Equation (ADE). Table 1 shows the relative error for $P_1$ and Table 2 shows that for $P_2$. From both the tables, Table 1 and Table 2, we can see that concentrations goes on decreasing with increase in distance. The relative error is less near the origin and more at far from the origin within the domain. The error for both pollution concentrations $P_1$ and $P_2$ is shown with the help of histogram below in Fig. 8, where red colored pillar shows the analytical solution whereas green colored pillar shows the numerical solution. As the analytical and numerical solutions for $P_1$ and $P_2$ are compared, the relative error is higher in computing $P_2$ rather than $P_1$, which can be seen clearly from the tables given below.

Table 1: Relative Error of pollution concentration $P_1$ [Kg m$^{-3}$] at $t = 3$ hours.

| Distance (km) | Analytical solution | Numerical solution | Error | Error % |
|---|---|---|---|---|
| 0 | 2.301356 | 2.283520 | 0.007750 | 0.770 |
| 5 | 0.945251 | 0.930611 | 0.015400 | 1.548 |
| 10 | 0.409821 | 0.399460 | 0.025281 | 2.530 |
| 15 | 0.193824 | 0.183838 | 0.05152 | 5.152 |
| 20 | 0.104777 | 0.098491 | 0.05999 | 5.999 |

Table 2: Relative Error of pollution concentration $P_2$ [Kg m$^{-3}$] at $t = 3$ hours.

| Distance (km) | Analytical solution | Numerical solution | Error | Error % |
|---|---|---|---|---|
| 0 | 2.405330 | 2.427610 | 0.009260 | 0.926 |
| 5 | 1.089690 | 1.121450 | 0.029140 | 2.914 |
| 10 | 0.507550 | 0.477210 | 0.059770 | 5.977 |
| 15 | 0.182539 | 0.169762 | 0.069990 | 6.990 |
| 20 | 0.085641 | 0.092935 | 0.085160 | 8.516 |



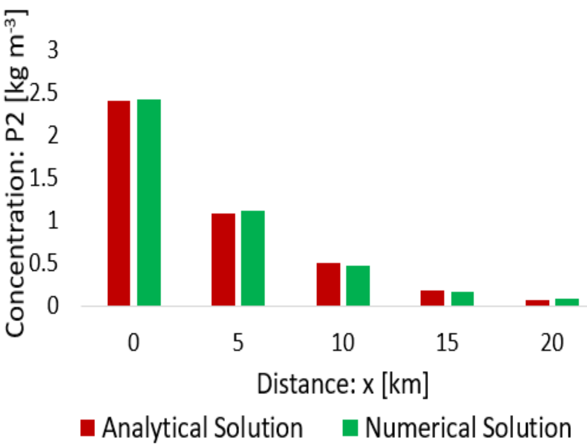




Figure 8: Comparison of analytical and numerical solutions for water pollution concentration with uniform (Left) and exponential (Right) increment of pollutants at origin. In both the cases analytical as well as numerical solutions are obtained close to each other.

## 4 Conclusion

In this study, one dimensional Advection-Dispersion Equation (ADE) is applied to study the dynamics of water pollution concentration at different spatial region along the length of river as the pollutants in the source increasing uniformly and exponentially. The analytical solutions of the employed model are presented by employing Laplace transform and steady state state case is studied. It has been found that concentration becomes a fixed positive constant when time and space both tends to infinity. Dispersion coefficient of pollution concentration is taken to be non zero. The results show that the pollution concentration at the source is relatively larger in the case of exponentially increment than that for uniformly increment. However, as the water flows substantially, these concentrations vary with small margin due to their advection and dispersion through the medium of water. The variations of concentrations $P_1$ and $P_2$ are presented along the length of the river with respect to the variation in rate of added pollutants, cross-sectional area of the river, water flow velocity. The results reveals that as the water flows downstream, the concentration decreases. However, their rate of decrement varies. As the rate of added pollutants increases, the concentration decreases slowly. Similarly, as velocity of the water increases, the water pollution advects rather than the dispersion that slows the decrease in concentration variation. When the area of cross section of the river channel increases, there will high discharge of water through the channel. Consequently, pollution concentration quickly decreases.

The three dimensional plots reveal that pollution concentration is the highest at origin (source) at the initial time and it decreases over time and space. However, it quickly decreases with traveled distance rather than traveled time due to the increment of pollutants either uniformly or exponentially. The numerical solutions of the model equations are obtained by Finite Difference Method (FDM) using Forward Time Central Space Scheme (FTCSS). As the analytical and numerical solutions for $P_1$ and $P_2$ are compared, the relative error is higher in computing $P_2$ rather than $P_1$. These results may be useful in better understanding the water pollution that may contribute to environmental engineering, hydrology, contaminant transport modeling and chemical engineering where the advection and dispersion of contaminants is widely applied.

## References

[1] P. Goyal and A. Kumar. Mathematical modeling of air pollutants: An application to Indian urban city. *Air Quality-Models and Applications*. doi:10.5772/16840, 2011.

[2] A. S. Wadi, M. F. Dimian, and F. N. Ibrahim. Analytical solution of one-dimensional advection-dispersion equation of the pollutant concentration. *Journal of Earth System Science*, 123(6):1317–1324, 2014.

[3] S. A. Alruman, A. F. El-kott, and M. A. Keshk. Water pollution: Source and treatment. *American Journal of Environmental Engineering*, 6(3):88–98, 2016.

[4] M. Haseena, M. F. Malik, A. Javed, S. Arshad, N. Adif, S. Zulqar, and J. Hanif. Water pollution and human health. *Environmental Risk Assessment and Remediation*, 1(3):16–19, 2017.

[5] N. Manitcharoen and B. Pimpunchat. Analytical and numerical solutions of pollution concentration with uniformly and exponentially increasing forms of sources. *Journal of Applied Mathematics*, 2020:1–9, 2020.

[6] B. Pimpunchat, W. L. Sweatman, W. Triampo, G. C. Wake, and A. Parshotam. Modelling river pollution and removal by aeration. *Modelling and Simulation Society of Australia and New Zealand*, pages 2431–2437, 2007.

[7] B. K. Mishra, R. K. Regmi, Y. Masago, K. Fukushi, P. Kumar, and C. Saraswat. Assessment of Bagmati river pollution in Kathmandu Valley: Scenario-based modeling and analysis for sustainable urban development. *Sustainability of Water Quality and Ecology*, 9:67–77, 2017.

[8] K. Pokhrel. *Water quality monitoring of Bagmati river basin, Kathmandu Valley.* Doctoral dissertation, Amrit Campus, 2023.

[9] M. T. Van Genuchten and W. J. Alves. Analytical solution of the one-dimensional convective-dispersive solute transport equation volume bulletin no.1661. *US Department of Agriculture, Agricultural Research Service*, 1982.

[10] K. Poudel, J. Kafle, and P. S. Bhandari. Analytical solution for advection-dispersion equation of the pollutant concentration using Laplace transformation. *Journal of Nepal Mathematical Society*, 4(1):33–40, 2021.

[11] G. A. L. Marusic. A study on the mathematical modeling of water quality in river type aquatic systems. *Journal WSEAS Transactions on Fluid Mechanics*, 8(2):80–89, 2013.

[12] N. Pochai, S. Tangmanee, L. J. Crane, and J. J. Miller. A mathematical model of water pollution control using the finite element method. *Proceedings in Applied Mathematics and Mechanics*, 6(1):755–756, 2006.

[13] P. Tabuenca, J. Vila, J. Cardona, and A. Samartin. Finite element simulation of dispersion in the Bay of Santander. *Advances in Engineering Software*, 28(5):313–332, 1997.

[14] H. Johari, N. Rusli, and Z. Yahya. Finite difference formulation for prediction of water pollution. *Materials Science and Engineering*, 318(012005):1–10, 2018.

[15] J. J. Miller, N. Pochai, S. Tangmanee, and L. J. Crane. A water quality computation in the uniform channel. *Journal of Interdisciplinary Mathematics*, 11(6):803–814, 2008.

[16] G. E. Roberts and H. Kaufman. *Table of Laplace transforms.* W. E. Saunders Co., Philadelphia and London, 1969.

[17] H. S. Carslaw and J. C. Jaeger. Conduction of heat in solids. *Clarendon Press*, Oxford, UK, 1959.

[18] S. Savović and A. Djordjevich. Finite difference solution of the one-dimensional advection–diffusion equation with variable coefficients in semi-infinite media. *International Journal of Heat and Mass Transfer*, 55(15-16):4291–4294, 2012.

[19] A. Kumar, D. K. Jaiswal, and N. Kumar. Analytical solutions to one-dimensional advection–diffusion equation with variable coefficients in semi-infinite media. *Journal of Hydrology*, 380(3-4):330–337, 2010.

[20] C. S. Chapra. Surface water-quality modeling. *The McGraw-Hill Companies*, 1997.

[21] W. E. Dobbins. BOD and oxygen relationships in streams. *Journal of the Sanitary Engineering Division*, 90(3):53–78, 1964.

[22] G. Li. *Stream temperature and dissolved oxygen modeling in the lower Flint River basin, GA.* Doctoral dissertation, University of Georgia, 2006.

[23] G. E. Roberts and H. Kaufman. *Table of Laplace transforms.* W. E. Saunders Co., Philadelphia and London, 1969.

[24] D. K. Salkuyeh. On the finite difference approximation to the convection-diffusion equation. *Applied Mathematics and Computation*, 179(1):79–86, 2006.

[25] R. K. Singh, H. Yabar, N. Nozaki, and R. Rakwal. Analyzing waste problems in developing countries: Lessons for Kathmandu, Nepal through analysis of the waste system in Tsukuba city, Japan. *Journal of Scientific Research and Reports*, 8(6):1–13, 2015.