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Caputo-Fabrizio approach to numerical fractional derivatives

Shankar Pariyar[1,2], Jeevan Kafle[2*]

[1] *Tri-Chandra Multiple Campus, Ghantaghar, Kathmandu, Nepal,*
[2] *Central Department of Mathematics, Tribhuvan university, Kathmandu, Nepal*

[*]Corresponding author.Jeevan Kafle,: jeevan.kafle@cdmath.tu.edu.np

Abstract

Fractional calculus is an essential tool in every area of science today. This work gives the quadratic interpolation-based L1-2 formula for the Caputo-Fabrizio derivative, a numerical technique for approximating the fractional derivative. To get quadratic and cubic convergence rates, respectively, we study the use of Lagrange interpolation in the L1 and L1-2 formulations. Our numerical analysis shows the accuracy of the theory's predicted convergence rates. The L1-2 formula aims to enhance the accuracy and usability of a flexible tool for many applications in science and mathematics. We demonstrate the validity of the theory's predicted convergence rates using numerical analysis. Several numerical examples are also given to show how the suggested approaches may be utilized to determine the Caputo-Fabrizio derivative of well-known functions. Lagrange interpolation is used in the L1 and L1-2 procedures to obtain quadratic and cubic convergence rates, respectively. The numerical study demonstrates that the L1-2 formula offers greater accuracy when compared to current approaches. In addition, it is a better apparatus for several applications in science and mathematics. Due to its higher convergence rate, the L1-2 formula outperforms other available numerical methods for scientific computations. The L1-2 formula, a novel numerical method for the Caputo-Fabrizio derivative that makes use of quadratic interpolation, is introduced in this study as a conclusion.

Keywords
Caputo fractional derivatives, Numerical solution, Lagrange interpolation, L1 formula, L1-2 formula.

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1 Introduction

A branch of mathematics known as numerical analysis for fractional calculus is concerned with the creation and use of numerical techniques for the solution of fractional calculus-related issues [1]. The study of fractional calculus is still in its infancy and is continually changing. Integrating and differentiating non-integer categories is the main concept behind it. It is extensively employed across a range of scientific and technological fields of study, including the sciences of physics, engineering, biology, economics, and others [1].

In contrast to fractional integral equations, which involve fractional integrals, fractional differential equations are defined by the presence of fractional derivatives. A system is said to have a differential order if its behavior can be correctly predicted by one of these equations, both of them, or both together [2]. Models based on fractional-order differential equations have been developed to describe complex dynamical systems and physical phenomena. To advance the theory and application of fractional order differentials, committed researchers have made substantial efforts. The fractional derivatives rule is not a collection of rules that can be applied in every situation, although the terms fractional differentiation and integration have been defined differently by various mathematicians [3]. Because Riemann-Liouville (R-L) shows that the derivative of a constant component is not zero, it has been demonstrated that it can be difficult to calculate fractional derivatives using regular calculus. The R-L fractional derivative is the meaning of these that is most widely used. Constant terms have non-zero derivatives, as demonstrated by the R-L discovery, which demonstrates that the derivative of a constant term is not zero, In the framework of classical calculus, it is difficult to analyze fractional derivatives [4, 5].

Jumarie expertly overcame this difficulty in the most recent version by modernizing the idea of F.D. of the R-L category. The fractional derivative versions of Caputo, Grunwald-Letinikov (G-L), and Jumarie's modified R-L are the most successful in avoiding the problem of non-zero derivatives resulting from a constant function [2]. While the Grunwald-Letinikov (G-L) term is useful for numerical applications, When it comes to analytical techniques, the concepts of R-L and Caputo are especially useful. No one technique can be used to solve the linear fractional differential equations. The derivative of the Mittag-Leffler function (MLF) [6] can be calculated using the fractional derivative with the Jumarie modification.

Swedish scientist ML developed the MLF for averaging divergent series at the beginning of the 20th century. Given its crucial part in supplying solutions with integral and derivatives of fractional order, MLF is a special transcendental function. The fractional order derivatives or integrations of the superdiffusively transport, random walk, kinetic equation, and complex systems are readily solved by the MLF [4, 7].

Both the standard and extended MLF interpolate phenomena are governed by shared kinetic equations and their fractional analogs between an exponential law and a power law [8, 9]. In contrast to linear fracti,onal differential equations, which cannot be solved by a single technique, the FD of the MLF function can be derived by using the updated definition of the FD proposed by Jumarie. The exponential function e^z is essential for integer-order differential equations, according to theory. G.M. Mittag-Leffler was the first to refer to the function that is now denoted by, with its one parameter generalization [10], was cited by G.M. Mittag-Leffler as the source first [7]. Fractional calculus can be used to solve fractional differential equations and explain odd phenomena. It is closely linked to MLF and its expansions. The two-parameter MLF, $R(\beta) > 0$, was established by Goreflo et al and Agarwal [11] immediately after the introduction of this version with a second complex parameter.

Leibniz asked L' Hospital [12] about the consequences of $n = \frac{1}{2}$ and the n^{th} derivative of the $f(t) = t$ linear model. FC was developed in response to this query. Leibniz discovered a paradox that had human consequences, but he also noted the creation of FC on September 30, 1695. A 34-year delay later, in 1849, he added the beta and gamma functions to it and referenced to G.H. 1849 [13]. Lagrange [14] developed and published the exponent principle for integer-order differential operators in 1772. Some of the most significant definitions of FC from several eminent mathematicians are given by us. Numerous mathematicians, including Fourier, Lacroix, Riemann, Liouville, Caputo, Euler, and G-L [15], were inspired by Romero's work and his citation by Laplace [16] in 1812 to advance the subject of fractional calculus. They also made use of Legendre's extended factorial notation and Lacroix's simple nth-order derivative [11].

By using the Laplace transform approach, Humbert and Agarwal [15] created a number of connections for this study. Solving oscillation and relaxation equations is governed by first and second orders of linear differential equations [17, 18]. Caputo and Fabrizio [19] in 2015 have introduced a new type of fractional derivative called CFD to remove the singular kernel. Using both Caputo-Fabrizio and Atangana-Baleanu derivatives, Algahtani [20] (2016) carried out a numerical comparison of the Allen-Cahn equation Gao et al. [13] in (2014) introduced a novel fractional numerical technique, the L1-2 formula, for the Caputo fractional derivative, which improved the accuracy of numerical solutions from $\delta t^{2-\alpha}$ to $\delta t^{3-\alpha}$ via quadratic interpolation. The aforementioned research served as an inspiration for the creation of a new quadratic Lagrange interpolation L1-2 formula. The accuracy of the Caputo-Fabrizio derivative (CFD) was shown to be able to be increased from quadratic convergence to cubic. To the best of our knowledge, no previous L1-2 differentiation formula for the Caputo-Fabrizio derivative (CFD) has been presented before this study. Additionally, this formula's convergence rate has been developed. Even though Caputo Fabrizio's numerical solution, which makes use of the L_1 and L_2 numerical formulae, is the main emphasis of this paper, there are other numerical solutions to some specific fractional calculus functions. Gao et al. [13], introduced an innovative fractional numerical method, the $L1-2$ formula, for the CFD and used nonlinear interpolation to increase the precision of the numerical calculation at different steps.

Our sounds show how the quadratic rate of convergence can be transformed into a cubic, even fourth-order, CFD. This study, in our opinion, is the first to support the convergence of the method and to give a new $L1-2$ differentiation formula for CFD. In addition to studying straightforward fractional relaxation and vibration equations, Gerenflo and Mainardi [11] also investigated fractional derivatives. From 1695 to 1697, Leibniz wrote correspondence to J. Wallis and J. Bernoulli [17,18] in which he offered a potential method for differentiating fractional orders [6].

2 Preliminaries

Some of the priliminaries which are needed for the work are given below :

Definition 1 *The formula below represents the gamma function's integral* $\Gamma(p)=\int_0^\infty e^{-x}x^{(p-1)}dx$, $p>0$. *Because only when* $p>0$ *does the aforementioned integral converge. The factorial function is also generalized by the gamma function, as for any positive integer* p, $\Gamma(p)=\int_0^\infty e^{-x}x^{(p-1)}dx=(p-1)!$ *[21].*

Definition 2 *The beta function, denoted as* $\beta(p,q)$ *with* $p,q>0$, *is defined as the integral of* $x^{p-1}(1-x)^{q-1}$ *from 0 to 1. It is related to the gamma function through the formula* $\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ *[22].*

Definition 3 *According to GL, the function* $M(x)$ *of* n^{th} *order FD with respect to* x *is defined by GL if* α *is a non-negative real number [2].*

GL Left sided FD: $${}^{GL}D^{\alpha}_{a+}[M(x)]=\lim_{n\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{n}(-1)^k\frac{\Gamma(\alpha+1)M(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)},nh=(x-a).$$

GL Right sided FD: $${}^{GL}D^{\alpha}_{b-}[M(x)]=\lim_{n\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{n}(-1)^k\frac{\Gamma(\alpha+1)M(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)},nh=(b-x).$$

Definition 4 *The R-L fractional Integral,*

$${}_a\boldsymbol{I}^{\alpha}_{x}(m(x))=\frac{1}{\Gamma(\alpha)}\int_a^x(x-z)^{\alpha-1}m(t)dt;\ x>a.$$

$${}_x\boldsymbol{I}^{\alpha}_{a}(m(x))=\frac{1}{\Gamma(\alpha)}\int_x^a(x-z)^{\alpha-1}m(t)dt;\ \ a>x.$$

Properties

P1: ${}_aI^{\alpha}_x(K(T(x)))=K\cdot{}_aI^{\alpha}_x(T(x))$.

P2: ${}_aI^{\alpha}_x(P(x)\pm Q(x)))={}_aI^{\alpha}_x(P(x))\pm{}_aI^{\alpha}_x(Q(x))$.

P3: ${}_aI^{\alpha}_x(N(x)\pm M(x))={}_aI^{\alpha}_x(N(x))\pm{}_aI^{\alpha}_x(M(x))$.

Definition 5 *under Riemann-Liouville, in most cases, the Riemann-Liouville formula defines a fractional derivative [23].*

$${}_b\boldsymbol{D}^{q}_{t}(v(x))=\begin{cases}\frac{1}{\Gamma(\nu-\gamma)}\left(\frac{d}{dx}\right)^t\int_b^q(q-t)^{t-\gamma-1}v(t).dt;(t-1)<\beta<t\\ \frac{d^t}{dx^t}v(x),\quad\beta=t\end{cases}$$

Definition 6 *The Caputo Fractional derivative is commonly used [13].*

$${}_a\boldsymbol{D}^{\alpha}_{x}(m(x))=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\left(\frac{d}{dx}\right)^n m(t).dt,\\ \qquad (n-1)<\alpha<n\\ \frac{d^n}{dx^n}\cdot m(x);\quad\alpha=n\end{cases}\tag{1}$$

Definition 7 *Mittag - Leffler Function (MLF) of one parameter [10],*

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha k+1)}\tag{2}$$

Two-parameters MLF β, $R(\beta)>0$ *[11] and Agarwal [24]* β, $R(\beta)>0$

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha\cdot k+\beta)}\quad(z,\alpha,\beta\in C)\tag{3}$$

The generation of 3 in terms of series representation was introduced by Prabhakar.

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_n z^k}{\Gamma(\alpha\cdot k+\beta)\cdot n!}\tag{4}$$

Where $(\gamma)_n$ *is Pochhammer's symbol. With the aforementioned considerations in mind, we discuss the fractional operators with three-parameter generalized Mittag-Leffler kernels in our work.*

Definition 8 *A novel fractional derivative with no singularities in its kernel was suggested by Caputo and Fabrizio in their [17]. The first fractional derivative has an exponential kernel, which is similar to an exponential function.* ${}^{CF}_{a}D_x(f(x))=\frac{M(\alpha)}{(1-\alpha)}\int_a^x e^{-\frac{\alpha(x-t)}{(1-\alpha)}}\cdot f'(t)\cdot dt;\ M(0)=M(1)=1$

Definition 9 *Interpolation: The process of figuring out the values of the functions at any intermediate point when the values of the first two points are known is known as interpolation.*

Definition 10 *Linear Interpolation [25]: The linear polynomial that passes through the specified coordinates* (a_0, b_0) *and* (a_1, b_1). *One method to express its formula is as follows:*
$P_1(x) = b_0 \frac{a_1 - x}{a_1 - a_0} + b_1 \frac{x - a_0}{a_1 - a_0}$

Definition 11 *Quadratic Interpolation [25]: The quadratic polynomial that passes through the specified coordinates* (a_0, b_0), (a_1, b_1), *and* (a_2, b_2) *may be written as*

$$P_2(x) = b_0 M_0(x) + b_1 M_1(x) + b_2 M_2(x),$$
$$M_0(x) = \frac{(x - a_1)(x - a_2)}{(a_0 - a_1)(a_0 - a_2)},$$
$$M_1(x) = \frac{(x - a_0)(x - a_2)}{(a_1 - a_0)(a_1 - a_2)}$$
$$M_2(x) = \frac{(x - a_0)(x - a_1)}{(a_2 - a_0)(a_2 - a_1)}$$

In the formula above, the polynomial $P(x)$ *is known as the Lagrange's interpolating polynomial, and* M_0, M_1, *and* M_2 *are the Lagrange's interpolating basis functions.*

3 Mathematical and numerical methods in Caputo-Fabrizio sense

3.1 L_1 Method for the Caputo Fractional Derivative

Left Caputo fractional derivatives are those fractional derivatives of a function that characterize its left-sided behavior, and the exact numerical method used to approximate them is called L_1 for $0 < \alpha < 1$

$$ {}_0^C\mathbf{D}_t^\alpha f(x) \approx \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s) ds \quad (5)$$
$$ {}_0^C\mathbf{D}_t^\alpha f(x_n) \approx \frac{1}{\Gamma(1-\alpha)} \int_0^{x_n} (x_n-s)^{-\alpha} f'(s) ds \quad (6)$$

$$ {}_0^C\mathbf{D}_t^\alpha f(x_n) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (x_n-s)^{-\alpha} f'(s) ds$$

Now we replace first-order derivative in 6 by forward difference quotient as follows:

$$ {}_0^C\mathbf{D}_t^\alpha f(x_n) \approx \frac{1}{h\Gamma(1-\alpha)} \sum_{k=0}^{n-1} [f(x_{k+1} - f(x_k)]$$
$$ {}_0^C\mathbf{D}_t^\alpha f(x_n) \approx \frac{h^\alpha}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] \; [(n-k)^{(1-\alpha)} - (n-k-1)^{(1-\alpha)}]$$

is known as the L_1 approach for Caputo Fractional Derivative. This method is the most popular way to approximate the Caputo fractional derivatives of a function. Using computer tools, we have since conducted a more thorough analysis of the numerical method and compare it to the precise solution.

3.2 L_2 Method for the Caputo Fractional Derivative

The left Caputo fractional derivative can be approximated via the L_2 approach

$$ {}_0^C\mathbf{D}_x^\alpha f(x) \approx \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{1-\alpha} f''(s) \cdot ds, \; (1 < \alpha < 2) \quad (7)$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad (8)$$

From (7) and (8) as follows,

$$ {}_0^C\mathbf{D}_x^\alpha f(x) \approx \frac{1}{\Gamma(2-\alpha)} \int_0^{x_n} (s)^{1-\alpha} f''(x_n - s) ds,$$
$$ {}_0^C\mathbf{D}_x^\alpha f(x_n) \approx \frac{1}{h^2(2-\alpha)\Gamma(2-\alpha)} (s)^{1-\alpha} \sum_{k=0}^{n-1} [f(x_n - x_{k+1}) - 2f(x_n - x_k) + f(x_n - x_{k-1})] \; [(x_{k+1})^{(2-\alpha)} - (x_k)^{(2-\alpha)}]$$

L_2 method for obtaining Caputo fractional derivatives is,

$$ {}_0^C\mathbf{D}_x^\alpha f(x_n) \approx \sum_{k=0}^{n-1} w_k f(x_{n-k}) \quad (9)$$

Such that where w_k is the normalizing factor and k has a range of values between -1 and n. It is generally acknowledged that the L_2 approach for Caputo Fractional Derivative is the preferred method for approximating fractional Caputo derivatives of functions. Recently, a more thorough analysis of this numerical method was done, and the results were compared to the exact ones.

3.3 L_1 Method for the Caputo - Fabrizio Derivative

The Caputo Fabrizio Differential operator is defined below for $0 < \alpha < 1$

$$ {}_0^{CF}\mathbf{D}_x^\alpha f(x) \approx \frac{M(\alpha)}{(1-\alpha)} \int_0^x f'(s) exp[\lambda(x-s)] ds, \lambda = \frac{-\alpha}{(1-\alpha)}, \quad (10)$$

$$ {}_0^{CF}\mathbf{D}_x^\alpha f(x_n) \approx \frac{M(\alpha)}{(1-\alpha)} \int_0^{x_n} f'(s) exp[\lambda(x_n-s)] ds, \quad (11)$$

$$ {}_0^{CF}\mathbf{D}_x^\alpha f(x_n) \approx \frac{M(\alpha)}{(1-\alpha)} \sum_{k=1}^{n} \int_{x_k-1}^{x_k} f'(s) exp[\lambda(x_n - s)]ds \tag{12} $$

Now, we replace first-order derivative in (13) by the forward difference quotient as follows:

$$ {}_0^{CF}\mathbf{D}_x^\alpha f(x_n) \approx \frac{M(\alpha)}{(1-\alpha)} \sum_{k=1}^{n} \int_{x_k-1}^{x_k} exp[\lambda(x_n - s)] \left[\frac{f(x_k) - f(x_{k-1})}{h}\right]ds \tag{13} $$

${}_0^{CF}\mathbf{D}_x^\alpha f(x_n) = \frac{M(\alpha)}{\alpha h} \sum_{k=1}^{n} [f(kh) - f(k-1)h]$ $[exp[\lambda h(n-k)] - exp[\lambda h(n-k+1)]$
which is a derivative of the Caputo Fabrizio equation using the L_1 Method. We demonstrate how the three-point backward estimate method, which we derive from the linear Lagrange method, was used to approximate the Caputo Fabrizio derivative. The cubic Lagrange interpolation can be represented with α range of $1 < \alpha < 2$.

3.4 L1-2 Method for Caputo Fractional Derivative

The CFD operator is defined below ${}_0^{CF}\mathbf{D}_y^\alpha g(y) = \frac{M(\alpha)}{(1-\alpha)} \int_0^y g'(u) exp[\lambda(y-u)]du, \lambda = \frac{-\alpha}{(1-\alpha)}, \alpha \in (0,1)$. ${}_0^{CF}\mathbf{D}_y^\alpha f(y_k) = \frac{M(\alpha)}{(1-\alpha)} [\int_{y_0}^{y_1} g'(u) exp[\lambda(y_k - u)]du + \int_{y_1}^{y_2} g'(u) exp[\lambda(y_k - u)]du + \cdots + \int_{y_{k-1}}^{y_k} g'(u) exp[\lambda(y_k - u)]du]$, Using the quadratic Lagrange interpolation with the points $(y_{j-2}, g_{j-2}), (y_{j-1}, g_{j-1}), (y_j, g_j)$ for the backward approximation first order derivative as follows,

$$ \begin{aligned} \Pi_{2,j} g(y) = & g_{j-2} \frac{(y - y_{j-1})(y - y_j)}{(y_{j-2} - y_{j-1})(y_{j-2} - y_j)} + \\ & g_{j-1} \frac{(y - y_{j-2})(y - y_j)}{(y_{j-1} - y_{j-2})(y_{j-1} - y_j)} + \\ & g_j \frac{(y - y_{j-2})(y - y_{j-1})}{(y_j - y_{j-2})(y_j - y_{j-1})} \end{aligned} $$

Where $t \in [t_{j-1}, t_j]$ for $2 \le j \le k$ and
$y_j = jh, y_{j-1} = (j-1)h$.

$$ \begin{aligned} \Pi_{2,j}(g(y)) = & \frac{g_{j-2}}{2h^2}[(y - y_{j-1})(y - y_j)] + \\ & \frac{f_{j-1}}{h}[(x - x_{j-2})(x_j - x) + \frac{g_j}{2h^2} \\ & [(y - y_{j-2})(y - y_{j-1})] \end{aligned} $$

$$ \Pi'_2 j(g(y)) = \frac{g_{j-2}}{2h^2}[(2y - y_{j-1} - y_j)] + \frac{g_{j-1}}{h} [-2y - y_{j-2} + y_j] + \frac{f_j}{2h^2}[2y - y_{j-2} - y_{j-1}] $$

$$ \begin{aligned} {}_0^{CF}\mathbf{D}_y^\alpha g(y_k) \approx & \frac{(g_1 - g_0)}{\alpha h}(\delta_{1,k} - \delta_{0,k}) \\ & + \frac{1}{\alpha} \sum_{j=2}^{k} \Bigg[\frac{g_{j-2}}{2h}(1-2j) + \frac{g_{j-1}}{h}(2j-2) \\ & + \frac{g_j}{2h}(3-2j)(\delta_{j,k} - \delta_{j-1,k}) + \\ & \frac{g_{j-2} - 2g_{j-1} + g_j)}{h^2}(y_j \delta_{j,k}) - \frac{(1-\alpha)}{\alpha}\delta_{j,k} \\ & - y_{j-1}\delta_{j-1,k} + \frac{1-\alpha}{\alpha}(\delta_{j-1,k}) \Bigg] \end{aligned} $$

4 Test Examples,

Relation between the function $f(x) = e^{-x}$ and its L1 fractional derivative $D^{0.5} f(x)$

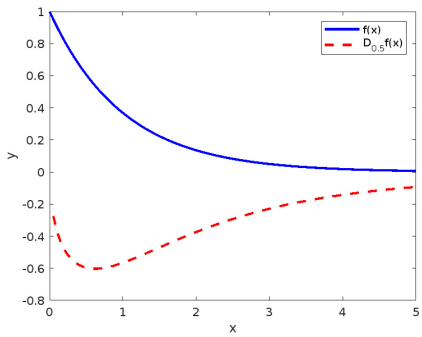

Figure 1: L_1 Fractional Derivative, $D^{0.5} f(x)$

A continuous blue curve for the function $f(x) = e^{-x}$ and a dashed red curve for $D^{0.5} f(x)$ will be displayed on the same plot in the two pictures. The function used in the instance, $f(x) = e^{-x}$, displays an exponential decline. The shape of $f(x)$ is determined by the selected function, $f(x) = e^{-x}$. The first derivative of the L_1 FD, which determines the rate of change, is standardized. Because the fact that the L_1 FD shape uses the complete history of the function and is a non-local operator, it is typically smoother and less noisy than the first derivative. The L1 FD is thought to be less that are susceptible to outliers and data noise than other varieties of FD.

Example 1 *Using the L1 formula and the equation f(x) = sin(x) over [0, 1] at* $\alpha = 0.5$*, the numerical results are shown in the table below.*

Table 1: L1 formula of $f(x) = sin(x)$ at $\alpha = 0.5$.

L1CD*	Exact	Error	Step Sizes
0.8559	0.8461	0.0099	h=0.1
0.8472	0.8461	0.0012	h=0.01
0.8462	0.8461	0.000111	h=0.001
0.8461	0.8461	0.00001074	h=0.0001
0.8461	0.8461	0.0000010611	h=0.00001

*L1CD represents L1-Caputo-Derivative.

Table 1 displays the numerical outcomes under schemes 5 for the function f(x) = sin(x) over the range [0, 1] using the L1-Caputo-Der formula. There are four columns in the table

- L1-Cap-Der: Using the L1-Caputo-Der formula, this column represents a numerical approximation of the integral of the function f(x).
- Exact: The value 0.8461 is shown in this column as the integral of the function f(x) over [0, 1].
- Inaccuracy: The absolute inaccuracy between the numerical approximation and the L1-Caputo-Der formula is shown in this column.
- Step Sizes: The fourth column labeled "Step Sizes" shows the step size used for each numerical result.

From the table, we can see that as the step size decreases, the numerical result obtained using the L1-Caputo-Der formula gets closer to the exact value of the integral. The error decreases as the step size decreases, indicating that the numerical approximation is becoming more accurate. At h=0.0001, the error is less than 0.00001, which is a very small error, indicating a highly accurate numerical approximation.

Example 2 *Using the L_2 formula and the equation f(x) = sin(x) over [0, 1] at* $\alpha = 1.5$*, the numerical results are shown in the table below.*

Table 2: L2 formula of f(x) = sin(x) at $\alpha = 1.5$

L2-CD*	Exact	Error	Step Sizes
-0.6219	-0.6697	0.0478	h=0.1
-0.6653	-0.6697	0.0044	h=0.01
-0.6693	-0.6697	0.00042716	h=0.001
-0.6696	-0.6697	0.000042431	h=0.0001
-0.6697	-0.6697	0.0000042307	h=0.00001

L2-CD* represents L2-Caputo Derivative.

Table 2 indicates that for the different step sizes to approximate the function $f(x) = sin(x)$ using the L2-Caputo-Derivative, using the scheme 9 over the range $[0, 1]$, $\alpha = 1.5$ is the value that was utilized in the approximation. The step size has an impact on the numerical precision. A smaller step size often results in a more precise estimate, even though it takes longer to calculate. A larger step size results in a less accurate approximation even though it takes less time to compute. As a result, while choosing a step size, accuracy and computation effectiveness must be compromised.

Example 3 *The equation f(x)=cos(4x) over the interval [0,2], with alpha=0.1 and different step sizes, can be solved using the L1-formula in the Caputo Fabrizio sense.*

Table 3: L1 for CFD at $f(x) = cos(4x)$ at $\alpha = 0.1$

ST*.	S.S*.	Ex*	Ax*	Er*
20×10^1	10^{-1}	-1.0811	-1.0807	3.9720×10^{-4}
20×10^2	10^{-2}	-1.0811	-1.0811	3.9720×10^{-6}
20×10^3	10^{-3}	-1.0811	-1.0811	3.9720×10^{-8}
20×10^4	10^{-4}	-1.0811	-1.0811	3.9720×10^{-10}

Acronyms:
ST* = Steps, S.S* = Steps Sizes, Ex* = Exact, Ax* = Approximate, Er*=Error.
If we observe an absolute error reduction of two orders of magnitude while simultaneously reducing the step size by one order of magnitude, this suggests a quadratic rate of convergence at the second order. In other words, not only is our progress significant but it is also happening quite rapidly.

Example 4 *The equation $f(x) = cos(4x)$ over the interval $[0, 2]$, with $\alpha = 0.1$ and different step sizes, can be solved using the $L1-2$(Backward Approximation) for Caputo-Fractional Derivativeformula in the Caputo Fabrizio sense.*

Table 4: L1-2 for CFD at $f(x) = cos(4x)$ at $\alpha = 0.1$

ST*.	S.S*	Ex*	Ax*	Er*.
2×10^1	1×10^{-1}	-1.0811	-1.0811	2.9514×10^{-7}
2×10^2	1×10^{-2}	-1.0811	-1.0811	2.9090×10^{-11}
4×10^1	0.05	-1.0811	-1.0811	1.8377×10^{-8}

If we observe an absolute error reduction of three orders of magnitude while simultaneously reducing the step size by one order of magnitude, this suggests a cubic rate of convergence at the third order. In other words, not only is our progress significant but it is also happening quite rapidly compared to L1-formula in the Caputo-Farbrizio sense.

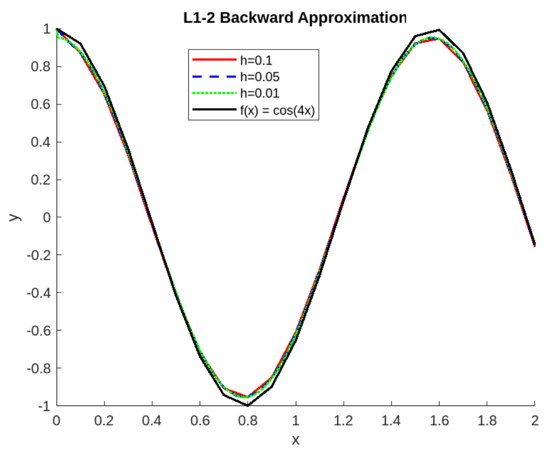

Figure 2: Your figure caption here.

When step sizes $h_1 = 0.1$, h_2= 0.05, and $h = 0.01$ are taken in the interval $[0, 2]$, an initial function $f(x) = cos(4x)$ is specified, with a curve to the function at each step size. The figure illustrates how, as step size h is reduced, the approximate solutions found using the $L1-2$ backwards approximate method approach the exact solution of the equation $f(x) = cos(4x)$.This is so that the function's derivatives can be approximated with greater accuracy using lower step sizes.

5 Comparative Analysis of a Fractional Derivatives

The Grunwald-Letnikov, Riemann-Liouville, Caputo, and L1-2 methods are examples of numerical methods for approximating fractional derivatives. Below is a brief comparison of these two methods:

Accuracy: when the fractional derivative's order is f and the step size is h the convergence rate of the Grunwald-Letnikov method is $O(h^{\alpha})$. The rate of convergence is not as rapid as other methods.

- Generally speaking, the Riemann-Liouville technique, which has a convergence rate of $O(h^2)$, is more accurate than the Grunwald-Letnikov method.
- The Caputo-Fabrizio technique is quicker than the Grunwald-Letnikov and Riemann-Liouville methods, with a convergence rate of $O(h^2)$.
- The quickest of all the approaches, the L1-2 Caputo method converges at a rate of $O(h^3)$.

Efficiency:

- The Grunwald-Letnikov approach only needs to evaluate a discrete convolution, hence it is computationally efficient.
- The Riemann-Liouville approach can be computationally demanding since it uses numerous integrals.
- Utilizing a finite difference formula, the Caputo-Fabrizio technique is computationally effective.
- To employ the computationally effective L1-2 Caputo approach, the quadratic interpolation formula must be assessed.

Stability:

- The Grunwald-Letnikov method could be numerically unstable depending on the values of the fractional derivative order and step size.
- The Riemann-Liouville method could be numerically unstable for particular values of the fractional derivative order and integration limitations.
- The Caputo-Fabrizio approach is often stable, however it can be sensitive to the selection of step size.
- Generally consistent and accurate with the L1-2 caputo approach.

Robustness:

- The Grunwald-Letnikov method is a usually reliable technique that can be used to calculate fractional derivatives for a variety of functions.
- The Riemann-Liouville approach is susceptible to the characteristics of the function being differentiated and can be constrained by the selection of integration limits.
- The Caputo-Fabrizio method can compute fractional derivatives for a variety of functions and is generally reliable.
- The L1-2 Caputo method is a usually reliable technique that can be used to calculate fractional derivatives for a variety of functions.

The L1-2 Caputo approach is the most precise, effective, stable, and reliable of the four ways, to put it briefly. Although less precise and effective than the L1-2 Caputo method, the Caputo-Fabrizio approach is nevertheless a viable alternative. Compared to Caputo-based approaches, the Grunwald-Letnikov and Riemann-Liouville methods are typically less precise, less effective, and can be vulnerable to numerical instabilities.

6 Conclusion

Our approach is different from most other domains. A unique numerical approach called as L1, L2, L1 has been devised using the linear and cubic interpolations for the numerical solution of the CFD. It uses the Caputo-Fabrizio sense and L1-2 formula. For backward/forward approximations, The fourth-order convergence rate is found for backward approximations, while the L1 and L1-2 equations provide quadratic and cubic convergence rates. The numerical outcomes were consistent with a well-known functions for different orders and varied step sizes. Every one of these approaches has advantages and disadvantages. In contrast to the Riemann-Liouville technique, which can be challenging to compute because of singularities, the Grünwald-Letnikov approach is simple to construct but can be slow. The L1-2 method is a promising new approach with a greater convergence rate, in contrast to the widely used Caputo method, which can have accuracy problems. The particular application and the trade-offs between precision, effectiveness, and simplicity will determine the strategy to choose. Investigating the L1-2 formula's stability and resilience in the presence of noise or other error sources might be beneficial. This can entail evaluating the formula's response to various noise or disturbance levels and contrasting its effectiveness with that of alternative numerical techniques.

References

[1] S. Pariyar and J. Kafle. Approximation solutions for solving some special functions of fractional calculus via caputo-fabrizio sense. *Nepal Journal of Mathematical Sciences (NJMS)*, 3(2):71–80, 2022.

[2] H.T. Davis. *The Theory of Linear Operators.* Myers Press, Bloomington, USA, 1985.

[3] T.V. Aleroev and S.V. Eronkhin. Parameteric identification of the fractional-derivative order in the bagley-torvik model. *Math. Models Comp. Simul.*, 2:219–225, 2019.

[4] S. Das. *Functional fractional calculus.* Springer-Verlag., 2nd edition, 2011.

[5] Y. Luchko. General fractional integrals and derivatives of arbitrary order. *Symmetry*, 13(5):755, 2021.

[6] G. Jumarie. On the solution of the stochastic differential equation of exponential growth driven by fractional brownian motion. *Applied Mathematics Letters*, 18(7):817–826, 2005.

[7] G Mittag-Leffler. Sur la nouvelle fonction e_α (x). *Comptes Rendus de l'Academie des Sciences Paris*, 137:554–558, 1903.

[8] J. Kafle, Thakur B.K., and I. B. Bhandari. Iterative methods for transient analysis of series rlc circuit. *BIBECHANA*, 17(1):68–76, 2020.

[9] R.P. Agarwal. A propos d'une note de m.pierre humbert. *Séances Acad. Sci.*, 236(21):2031–2032, 1953.

[10] A. Fenander. *Higher Transcendental Functions.* McGraw-Hill, New York, 1955.

[11] R. Gorenflo and F. Mainardi. *Fractional Calculus: Integral and Differential Equations of Fractional Order.* Springer Verlag Wien and New York, 1997.

[12] R. Hilfer. *Applications of Fractional Calculus in Physics.* World Scientific Publishing Company, Singapore, 2000.

[13] G.H. Gao, Z.Z. Sun, and H.W. Zhang. A new fractional numerical differentiation formula to approximate the caputo fractional derivative and its applications. *Journal of Computational Physics*, 259:33–50, 2014.

[14] J. L. Lavoi, Thomas J. Osler, and R. Tremblay. Fractional derivatives and special functions. *SIAM Review*, 18(2):240–268, 1976.

[15] A. Erdelyi. *Tables of Integral Transforms.* McGraw-Hill, New York, 1954.

[16] L. G. Romero, G. D. Medina, N. R. Ojeda, and J. H. Pereira. A new alfa-integral laplace transform. *Asian Journal of Current Engineering and Maths*, pages 59–62, 2016.

[17] M. Caputo. Distributed order differential equations modelling, dielectric induction and diffusion. *Frac. Calc. Appl. Anal.*, 4:421–442, 2001.

[18] A. Olagunju and L. Joseph. Construction of functions for fractional derivatives using matlab. *Journal of Advances in Mathematics and Computer Science*, 36(6):1–10, 2021.

[19] M. Caputo and M. Fabrizio. A new definition of fractional derivative without singular kernel. *Progress in Fracture and Damage Mechanics: Special Issue on Fractional Differential Equations and Their Applications*, 1(2):1–13, 2015.

[20] O.J.J. Algahtani. Comparing the atangana–baleanu and caputo–fabrizio derivative with fractional order: Allen cahn model. *Chaos, Solitons and Fractals*, 89:552–559, 2016.

[21] S. Abdullah. Exact solutions for some fractional differential equations. *Advances in Mathematical Physics*, 2015:567842, 2015.

[22] A. Carpinteri. *Fractals and Fractional Calculus in Continuum Mechanics.* Springer, Wien, 1997.

[23] H. Bulut, H.M. Baskonus, and F.B.M. Belgacem. The analytical solution of some fractional ordinary differential equations by the sumudu transform method. *Abstr. Appl. Anal.*, pages 1–6, 2013.

[24] P. Agarwal and H. Irmak. Some comprehensive inequalities consisting of mittag-leffler type functions in the complex plane. *Mathematical Modelling and Natural Phenomena*, 12, 2017.

[25] H. S. Malvar, Li-Wei He, and R. Cutler. High-quality linear interpolation for demosaicing of bayer-patterned color images. In *2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, volume 3, pages iii–485. IEEE, 2004.