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Modeling and parameter analysis of deflection of a beam

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ABSTRACT

In this paper, we present the model equation of a beam when it applies compression forces on ends of the beam and carries a load. For the structural point of view, there should be a suitable model to understand the behavior under different conditions of loading of a beam. Mathematical modeling is the simulation of a physical structure or physical phenomenon by constructing suitable analytic and numerical solution. We analyze the deflections of the beam by taking different structures of beam. The structures of beam depend on the compression forces on beams with different beams with different weights. We observe the deflection by applying various compression forces at the ends of the beam. The influence of the effect of some parameters appeared in mathematical formulations such as area moment of inertia (I), Young's modulus (E), load (W) and compressive force (P) on deflection variation are investigated in this paper. We analyze the results that how compression forces affect the system. We use finite difference method to solve the model equation numerically. We analyze and compare the numerical result with analytic solution.

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1. Introduction

Beams are structural components subjected to transverse gravity or vertical loading. A beam can be straight or curved and with constant cross-section or tapered. There are various types of beam depending upon the nature of the material, length, width, depth, and external forces in different positions. To decode what type of beam to be used in a structure depends upon the loading system and its potential deflection. Deflection is the displacement of any point or part of beam from its original position, measured in y-direction [7]. The specific values of deflection for a given load can be found through differential equation [13]. In practice a factor of safety is generally considered and addressed in the model so as to compensate it. The deflection of a beam should be considered to make decision in real world problems. Generally, we have to know the magnitude and the position of the maximum deflection on a beam under various loading systems for a good engineering structure. The reactions and internal stresses of the beam can properly be estimated only with the proper knowledge of the deflection.

Various research activities on the deflection on a beam are carried out in the past decades. Conway [5] observed the effect of load which is centrally located between two supports. Schile and Sierakowski [12] solved the problem of bending of a thin, simply supported at two points. Bulte [3] modelled the deflection of a beam as fourth order ordinary differential equation. This was converted into a system of first order ordinary differential equation. The solutions were obtained numerically applying Runge-Kutta fourth order method, and were written in terms of elliptical functions whereas Chucheepsakul et al. [4] expressed the solution in integral functions, and solved numerically by applying the shooting-optimization method and the finite element method. This is one of the classical approaches to find the deflection, and widely used to large deflection beam bending problem. Banerjee et al. [1] proposed the Non-linear shooting and Adomian decomposition methods (ADM) to find the large deflection of a cantilever beam. Tari [14] presented Euler-Bernoulli boundary value problem, and its deflection solutions in terms of the loading parameters. He et al. [9] included first derivative term in Euler-Bernoulli equation, and solved the non-linear large deflection of a beam. In such cases, the solutions show nonlinearity in their deformation behaviour. Batista [2] used the general solution to derive an approximate formula that provides an explicit relation between the beam load and its midpoint deflection. Ghuku and Saha [7] solved the nonlinear governing equation numerically using Lagrangian approach, and compared the results with experimental work.

The study of the modelling of a beam and its parameter analysis plays significant role in the field of mechanical, aerospace and civil engineering. The knowledge of bending moment, shearing forces, rate of deformation and the deflection of a beam are useful in our daily life which we have encountered with structural elements such as the designing of civil engineering structures, machine and automobile frames, and aircraft components.

In this paper, the model equation of a beam is presented as considering very small slope in the beam. So, the model equation is presented here by omitting the first derivative term in Euler-Bernoulli equation. Further, the deflections of a beam are analyzed for the variation of applied forces at the ends, the different loads and the Young's modulus of the beam. The model equation are numerically solved by applying the finite difference method, and it is outlined that the errors with analytic results.

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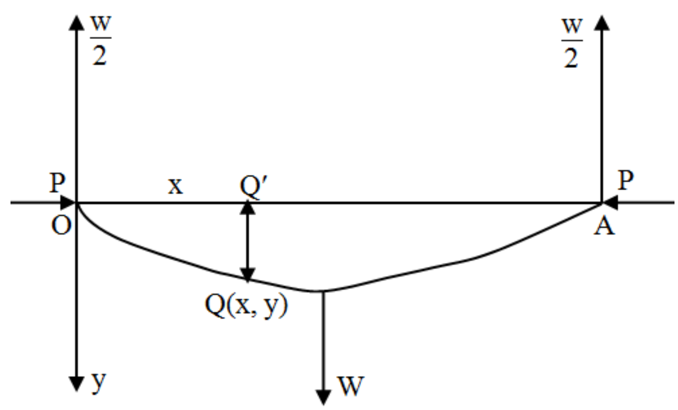

Figure 1.1: The deflection of a beam with compression forces.

2. Formulation of model equation

Let L be a length of a beam OA, □ be its radius of curvature of the beam at a distance x from origin, taken at the left hand end of the beam, after giving equal compressive force P at the ends O and B as shown in Fig.1. If slopes of deflection curve are small, then the slope of elastic curve is also small, i.e., $(dy/dx)^2$ is neglected in the formula [15]

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2},$$

where $y_1 = (d\,y\,/\,d\,x)$ and $y_2 = (d^2 y,\, d\,x^2)$, we get the radius of curvature for small deflection as $\rho = 1/\,(d^2 y\,/\,d\,x^2)$. Let M be the bending moment of the section at a distance x from the origin O, I be area the moment of inertia of the beam cross-section and E be the elastic or Young's modulus of the beam material from which the beam is made, then applying the Bernoulli - Euler law [1, 15] as $M = (E\,I/\,\rho)$ give

$$\mathrm{M} = E\,I\,\frac{d^2 y}{d\,x^2}, \qquad (1)$$

where y, x and M are variables, and E and I are constants. The bending moment M describes the amount of bending and deflection that occurs in a beam under a given loading system [6, 8]. It is defined as the sum of the moments of all forces to the left or right of the section that is under consideration. Let P be compressive force applied on both ends O and A as shown in Fig.1, the

$$\mathrm{M} = -\frac{W}{2}\,x - P\,y \qquad (2)$$

Then from (1) and (2), the model equation is

$$E\,I\,\frac{d^2 y}{d\,x^2} = -\frac{W}{2}\,x - P\,y$$

This is ordinary differential equation which can also be written as

$$(D^2 + n^2)\,y = -n^2\,k\,x, \qquad (3)$$

where $D^2 = (d^2/dx^2)$, $n^2 = P\,/EI$, $k = W\,/\,2P$. This is non-homogenous differential equation [11,13]. The general solution is $y\,(x) = y_c\,(x) + y_p\,(x)$, where $y_c\,(x)$ and $y_p\,(x)$ are complementary function and particular integral [11]. Let $y = e^{r\,x}$ be a solution of $(D^2 + n^2)\,y = 0$, then

$$e^{r\,x}\,(r^2 + n^2) = 0$$

Since $e^{r\,x} \neq 0$, $r^2 + n^2 = 0$, so that $r = \pm\, i\, n$, where $i = \sqrt{-1}$. So, complementary function [6] is $y_c\,(x) = c_1 \cos nx + c_2 \sin nx,$

and particular integral [7] is

$$y_p\,(x) = \frac{1}{D^2 + n^2}\left(-\frac{n^2 W}{2P}\,x\right)$$

$$= -\frac{n^2 W}{2P}\cdot\frac{1}{n^2}\left(1 + \frac{D^2}{n^2}\right)^{-1} x$$

$$= -\frac{W}{2P}\left(1 - \frac{D^2}{n^2} + \ldots\right) x = -\frac{W}{2P}\,x$$

General solution of (3) is $y\,(x) = y_c\,(x) + y_p\,(x)$ which gives

$$y\,(x) = c_1 \cos nx + c_2 \sin nx - \frac{W}{2P}\,x \qquad (4)$$

Applying the boundary conditions, $y = 0$ when $x = 0$, and $y = 0$ when $x = L$, we get

$$c_1 = 0 \text{ and } c_2 = \frac{W\,L}{2P \sin nL}.$$

Substituting the values of c_1 and c_2 in (4) gives

$$y\,(x) = \frac{W\,L}{2P \sin nL}\,\sin nx - \frac{W}{2P}\,x \qquad (5)$$

is analytic solution.

3. Results and discussion

We take the length of beam $L = 10$ m. Figure 1.2 shows that the maximum deflection is 4.404 mm at 5.45 m the length of beam when it is under the action of equal and opposite compressive forces $P = 65$ KN at its ends and it carries a load $W = 100$ Kg, Young's modulus $E = 100$ GPa and the area moment of inertia $I = 3$ cm^2. If the applied forces at the ends are reduced by 5 KN, then the maximum deflection of the beam are reduced at the length of beam $L = 5.45$ m.

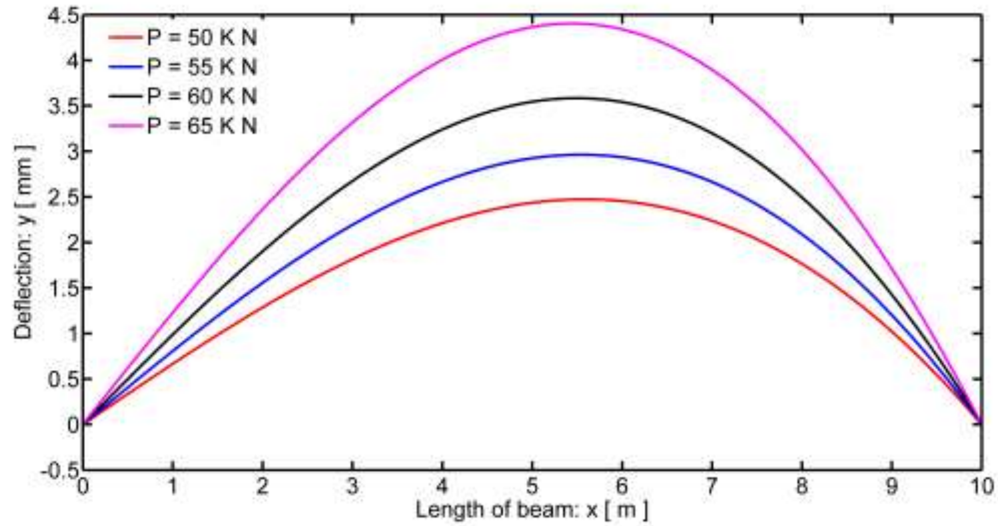

Figure 1.2: Deflection of beam with weight W = 100 Kg.

The maximum deflection are 3.5842 mm, 2.9635 m, 2.4732 mm when the applied forces at the ends are P = 60 KN, P = 55 KN, P = 50 KN respectively, as shown in Fig. 1.2. In the case of same compressive force, Young's modulus, and the area moment of inertia, when the loads of the beam are reduced by 10 Kg, then the maximum deflection of the beam is also reduced by 0.4404 mm. When the beam carries loads W = 90 Kg, W = 80 Kg, W = 70 Kg, then the maximum deflection at the length of beam L = 5.45 m are 3.9636 mm, 3.5232 mm, and 3.0828 mm respectively, as shown in Fig. 1.3. In the case of same compressive force, load, and the area moment of inertia, when there are different types of material on the beam so that Young's modulus of the beam are increased by 5 Gpa, then the maximum deflection of the beam are decreased by 0.4404 mm. When Young's modulus E = 105 GPa, E = 110 GPa, E = 115 GPa, then the maximum deflection at length of the beam L = 5.45 m are 3.6838 mm, 3.1446 mm, and 2.7271 mm respectively, as shown in Fig. 1.4. The maximum deflection is inversely proportional to the Young's modulus. If the ratio of stress and strain is increasing, then the maximum deflection is decreasing which is physically meaningful.

4. Numerical method

Consider the interval $R = \{x : 0 \le x \le L\}$, and it is subdivided into sub-interval with step size $\Delta x = h$ where $h = (L - 0) / (N + 1)$. The ends points are the mesh points $x_i = x_{i-1} + i\,h$, for $i = 1, 2, \ldots N$.

We choose the step size h in such a way that it facilitates the application of matrix. The second order derivative is approximated [10,11,13] as

$$\frac{d^2y}{dx^2} \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}.$$

Writing $W / 2P = k$, $n^2 = P / EI$, then the model equation (3) reduces to

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + n^2 y_i = -n^2 k\, x_i.$$

The finite difference numerical scheme is

$$y_{i-1} + (n^2 h^2 - 2)\, y_i + y_{i+1} = -n^2 k\, h^2 x_i,$$

for i = 1, 2, …,*N*.

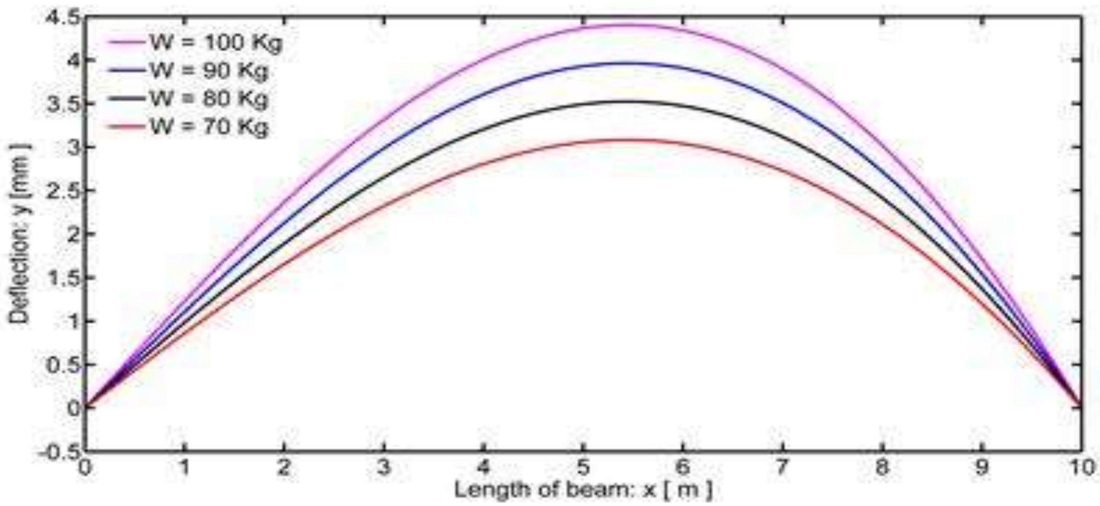

Figure 1.3: Deflection of a beam with compression force P = 65 K N.

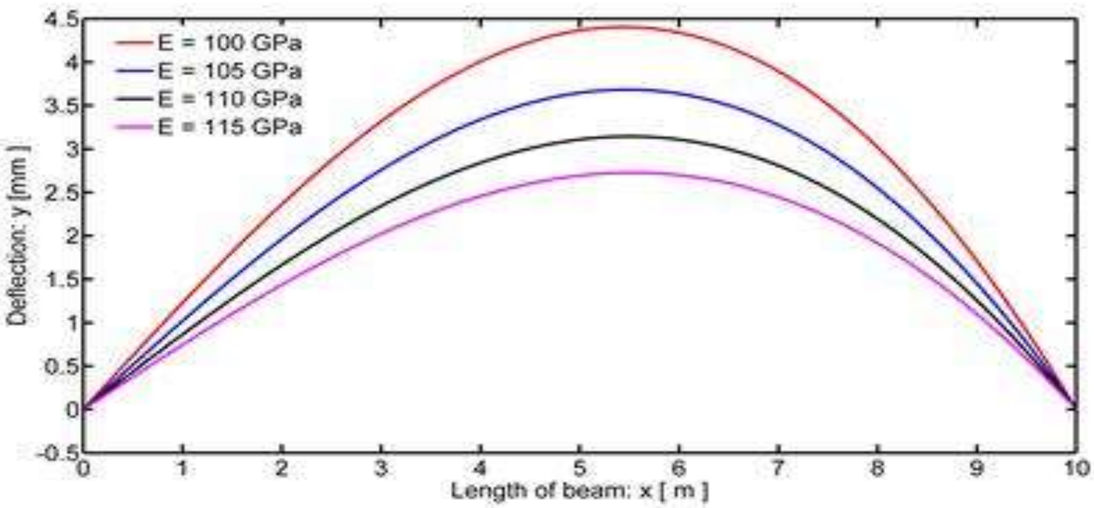

Figure 1.4: Deflection of beam with compression force P = 65 KN and load W = 100 Kg.

For i = 1 and applying the boundary conditions $y(0) = 0$ and $y(L) = 0$, we get

$$y_0 + (n^2 h^2 - 2) y_1 + y_2 = - n^2 k\, h^2 x_1,$$

$$\therefore \quad (n^2 h^2 - 2) y_1 + y_2 = - n^2 k\, h^2 x_1.$$

For i = 2,

$$y_1 + (n^2 h^2 - 2) y_2 + y_3 = - n^2 k\, h^2 x_2.$$

For i = 3,

$$y_2 + (n^2 h^2 - 2) y_3 + y_4 = - n^2 k\, h^2 x_3.$$

For i = N,

$$y_{N-1} + (n^2 h^2 - 2) y_N + y_{N+1} = - n^2 k\, h^2 x_N,$$

$$\therefore \quad y_{N-1} + (n^2 h^2 - 2) y_N = - n^2 k\, h^2 x_N - y_{N+1}$$

$$= - n^2 k\, h^2 x_N.$$

The system so formed, can be written in matrix form as

$$\begin{bmatrix} n^2h^2-2 & 1 & 0 & \dots & 0 \\ 1 & n^2h^2-2 & 1 & \dots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & n^2h^2-2 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} -n^2kh^2x_1 \\ -n^2kh^2x_2 \\ -n^2kh^2x_3 \\ \vdots \\ \vdots \\ -n^2kh^2x_{N-1} \\ -n^2kh^2x_N \end{bmatrix}$$

Table 1.1: Distribution of deflection of the beam with compression force P = 65 KN and load W = 100 Kg.

Length: L [m]	Defl.: y [mm] Numerical	Defl.: y [mm] Analytical	Error (ε)
0	0	0	0
0.8333	1.0519	1.0279	0.024
1.6667	2.0562	2.0017	0.0545
2.5000	2.9562	2.8686	0.0876
3.3333	3.6985	3.5797	0.1188
4.1667	4.2353	4.0912	0.1441
5.0000	4.5257	4.3654	0.1603
5.8333	4.5377	4.3726	0.1651
6.6667	4.2485	4.0917	0.1568
7.5000	3.6461	3.5112	0.1349
8.3333	2.7291	2.6289	0.1002
9.1667	1.5070	1.4526	0.0544
10.000	0	0	0

This system is solved here by using a computing program.

5. Numerical results

Figure 1.5 shows that numerical and analytical solution of the model equation. A beam of length L = 10 m, it carries a load W = 100 Kg, Young's modulus E = 100 GPa, the area moment of inertia, I = 3 cm^2 and compressive force at the ends of the beam P = 65 KN. The maximum deflection of the beam lies at x = 5.8333 m and decreases towards the extremes as shown in Fig.1.5. The maximum deflection substantially increases from 3 mm to 4.5377 mm. The distribution of errors of the deflection of the beam obtained from numerical and analytic method is shown in Table 1.1. The maximum error is 0.1651 and

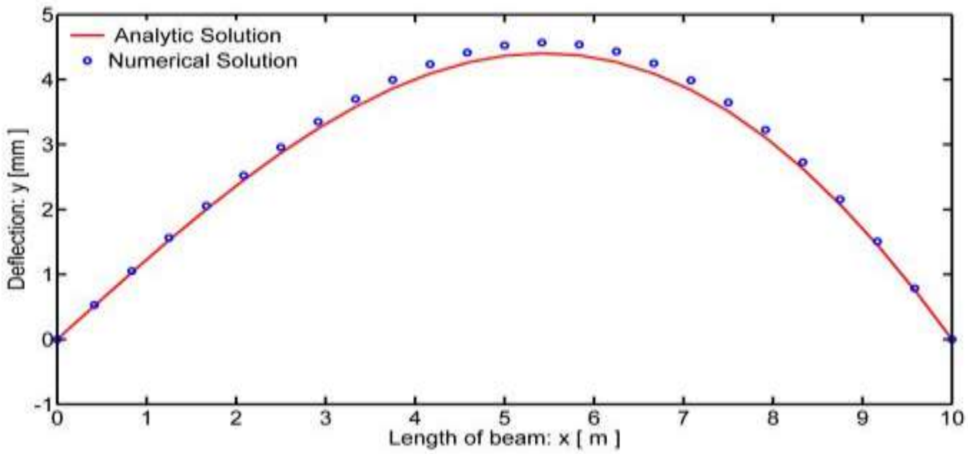

Figure 1.5: Deflection of beam with compression force $P = 65$ KN, load $W = 100$ Kg, Young's modulus $E = 100$ GPa, the area moment of inertia $I = 3$ cm^2.

the beam is highly deflected at the length of beam x = 5.8333 m. We see that the load of the beam plays more significant role than the horizontal for the deflection of the beam. The error in numerical solution of the deflection at the length of beam 3.3333m to 8.333m is more than from analytic solution.

6. Conclusion

We presented the model equation of a beam with external forces at the ends when it carries load, the area moment of inertia with material properties. We found that the variation of maximum deflection by applying the different external forces. The analytic and numerical solution of the deflection of the beam showed that the deflection depend on external forces as well as when it carries loads, the area moment of inertia and the structures. We analyzed the deflection of the beam for different horizontal compression forces from the extremities. The maximum deflection of the beam occurs at the centre of the beam and gradually decreases to zero at the extremities. The weight of the beam has major role for the deflection of the beam. We also analyzed the maximum deflection of the beam with different Young's modulus. We observed that error analysis of the solution of model equation, and we found that there was less error in the deflection of the beam between analytic and numerical results. The results can be used to estimate the failure of a beam.

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