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Ricci solitons on Lorentzian para-Sasakian manifolds

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ABSTRACT

In this paper we study Ricci solitons in Lorentzian para-Sasakian manifolds. It is proved that the Ricci soliton in a $(2n+1)$-dimensinal LP-Sasakian manifold is shrinking. It is also shown that Ricci solitons in an LP-Sasakian manifold satisfying the derivation conditions $R(\xi, X).W_2 = 0, W_2(\xi, X).W_4 = 0$ and $W_4(\xi, X).W_2 = 0$ are shrinking but are steady for the condition $W_2(\xi, X).S = 0.$ Finally, we give an example of 3-dimensional LP-Sasakian manifold and prove that the Ricci soliton is expanding and shrinking in this manifold.

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1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0, \qquad (1.1)$$

where S is the Ricci tensor and $L_V g$ denotes the Lie derivative of g along a vector field V [1]. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$ respectively. Compact Ricci solitons are the fixed points of the Ricci flow

$$\frac{\partial g}{\partial t} = -2S$$

projected from the space of metrics onto its quotient modulo diffeomorphism and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds.

Metrics satisfying (1.1) are interesting and useful in physics and often are referred as quasi-Einstein (eg, see [2], [3]). The Ricci flow was used by Perelman to prove the Poincare's conjecture theorem and the Thurston's geometrization conjecture theorem in topology [4]. Ricci solitons have also been studied by [5], [6], [7], [8] and others.

On the other hand, the notion of a Lorentzian para-Sasakian manifold was introduced by

Matsumoto [9]. Mihai and Rosca defined the same notion independently and obtain several results on this manifold [10]. LP-Sasakian manifolds have also been studied by [11], [12], [13] and others. In this paper, we prove some derivation conditions for Ricci solitons in LP-Sasakian manifolds. We investigate shrinking property of Ricci soliton in a LP-Sasakin manifold when a vector field V is collinear with ξ. We obtain some results of Ricci solitons on LP-Sasakian manifolds satisfying the conditions $R(\xi, X).W_2 = 0$, $W_2(\xi, X).S = 0$, $W_2(\xi, X).W_4 = 0$ and $W_4(\xi, X).W_2 = 0$ respectively. Finally, we give an example of 3-dimensional LP-Sasakian manifold which is expanding and shrinking Ricci soliton.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold. Then M is said to be a Lorentzian para-Sasakian manifold (briefly LP-Sasakian manifold), if it admits a (1, 1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \varphi^2(X) = X + \eta(X)\xi, \quad \varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$\nabla_X \xi = \varphi X, \tag{2.4}$$

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

for all $X, Y \in TM$, where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g [9, 10].

If we put,

$$\Phi(X, Y) = g(X, \varphi Y),$$

then Φ is a symmetric (0,2) tensor field [9]. Since the 1-form η is closed in an LP-Sasakian manifold we have [12], [9]

$$(\nabla_X \eta)Y = \Phi(X, Y) = g(X, \varphi Y) = g(\varphi X, Y). \tag{2.6}$$

In a $(2n+1)$-dimensional LP-Sasakian manifold the following relations hold

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.7}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$S(X, \xi) = 2n\eta(X), \tag{2.10}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

for any vector fields X, Y, Z, where R and S are the Riemannian curvature tensor and the Ricci tensor of the manifold, respectively [11].

Let (g, V, λ) be a Ricci soliton in a $(2n+1)$-dimensional LP-Sasakian manifold M. Then we have

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X).$$

Using (2.4) and (2.6) in this equation we get

$$(L_\xi g)(X, Y) = 2g(X, \varphi Y). \tag{2.12}$$

From (1.1) and (2.12) we obtain

$$S(X, Y) = -\{\lambda g(X, Y) + g(X, \varphi Y)\}, \tag{2.13}$$

$$r = -(2n+1)\lambda, \text{ provided } tr.\varphi = 0. \tag{2.14}$$

In view of (2.1), (2.3) and (2.13) we get

$$S(X, \xi) = -\lambda\eta(X). \tag{2.15}$$

3. Results and Discussion

Now, we have the following results and their proofs

Theorem 3.1: If in a $(2n+1)$-dimensional LP-Sasakian manifold the metric g is a Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is shrinking.

Proof: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold with Lorentzian metric g. A Ricci soliton is a generalization of an Einstein metric and defined on a Riemannian manifold (M, g) by (1.1). Let V be pointwise collinear with ξ i.e., $V = c\xi$ where c is a function on a $(2n+1)$-dimensional LP-Sasakian mani-fold. Then from (1.1), we have

$$(L_{c\xi}g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0. \quad (3.1)$$

Further simplification and use of (2.4) in (3.1) yields

$$\begin{aligned}&cg(\varphi X,Y)+(Xc)\eta(Y)+cg(\varphi Y,X)\\&+(Yc)\eta(X)+2S(X,Y)+2\lambda g(X,Y)=0.\end{aligned} \quad (3.2)$$

By virtue of (2.6) and (3.2) we obtain

$$\begin{aligned}&2cg(X,\varphi Y)+(Xc)\eta(Y)+(Yc)\eta(X)\\&+2S(X,Y)+2\lambda g(X,Y)=0.\end{aligned} \quad (3.3)$$

Putting $Y=\xi$ in (3.3) and using (2.1), (2.3) and (2.10) we get

$$\{(\xi c)+4n+2\lambda\}\eta(X)-(Xc)=0. \quad (3.4)$$

Taking $X=\xi$ in (3.4) gives

$$\xi c=-(2n+\lambda). \quad (3.5)$$

In view of (3.4) and (3.5) we obtain

$$Xc=(2n+\lambda)\eta(X),$$

which implies

$$dc=(2n+\lambda)\eta. \quad (3.6)$$

Taking exterior derivative on both sides of (3.6) we get

$$(2n+\lambda)d\eta=0, \quad (3.7)$$

since $d\eta\neq 0$, we have $2n+\lambda=0$. Hence c is constant from (3.6). Consequently, the equation (3.3) reduces to

$$S(X,Y)=-\{\lambda g(X,Y)+cg(X,\varphi Y)\}. \quad (3.8)$$

Comparing (2.13) and (3.8) we get $c=1$. Again, $2n+\lambda=0$ implies that $\lambda=-2n<0$ for $n>1$. Thus the Ricci soliton is shrinking. This proves the theorem.

Theorem 3.2: A Ricci soliton in a W_2-semi-symmetric LP-Sasakian manifold of dimension $(2n+1)$ is shrinking.

Proof: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold admitting a Ricci soliton (g,V,λ). The W_2-curvature tensor in M is defined by [14]

$$\begin{aligned}W_2(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{2n}[&g(X,Z)Ric(Y,T)\\&-g(Y,Z)Ric(X,T)],\end{aligned}$$

this can be written as

$$\begin{aligned}W_2(X,Y)Z=R(X,Y)Z+\frac{1}{2n}[&g(X,Z)QY\\&-g(Y,Z)QX].\end{aligned} \quad (3.9)$$

Putting $X=\xi$ in (3.9) and using (2.3) and (2.8) we get

$$\begin{aligned}W_2(\xi,Y)Z=&\,g(Y,Z)\xi-\eta(Z)Y\\&+\frac{1}{2n}[\eta(Z)QY-g(Y,Z)Q\xi].\end{aligned} \quad (3.10)$$

Taking inner product on both sides of (3.9) with ξ and using (2.7) and (2.15) we obtain

$$\begin{aligned}\eta(W_2(X,Y)Z)=\left(1+\frac{\lambda}{2n}\right)[&g(Y,Z)\eta(X)\\&-g(X,Z)\eta(Y)].\end{aligned} \quad (3.11)$$

Suppose that the condition $R(\xi,X).W_2(Y,Z)U=0$ holds in M. Then by definition we have

$$\begin{aligned}&R(\xi,X)W_2(Y,Z)U-W_2(R(\xi,X)Y,Z)U\\&-W_2(Y,R(\xi,X)Z)U-W_2(Y,Z)R(\xi,X)U\\&=0\end{aligned} \quad (3.12)$$

for all vector fields X,Y,Z,U on M.

In view of (2.8) and (3.12) we get

$$\begin{aligned}&g(X,W_2(Y,Z)U)\xi-\eta(W_2(Y,Z)U)X\\&-g(X,Y)W_2(\xi,Z)U+\eta(Y)W_2(X,Z)U\\&-g(X,Z)W_2(Y,\xi)U+\eta(Z)W_2(Y,X)U\\&-g(X,U)W_2(Y,Z)\xi+\eta(U)W_2(Y,Z)X\\&=0.\end{aligned} \quad (3.13)$$

Taking inner product on both sides of (3.13) with ξ and using (2.1) we obtain

$$\begin{aligned}&g(W_2(Y,Z)U,X)+\eta(W_2(Y,Z)U)\eta(X)\\&+g(X,Y)\eta(W_2(\xi,Z)U)-\eta(Y)\eta(W_2(X,Z)U)\\&+g(X,Z)\eta(W_2(Y,\xi)U)-\eta(Z)\eta(W_2(Y,X)U)\\&+g(X,U)\eta(W_2(Y,Z)\xi)-\eta(U)\eta(W_2(Y,Z)X)\\&=0.\end{aligned} \quad (314)$$

In view of (3.9), (3.11) and (3.14), we get

$$\begin{aligned}&g(R(Y,Z)U,X)+\frac{1}{2n}[g(Y,U)S(X,Z)\\&-g(Z,U)S(X,Y)]+\left(1+\frac{\lambda}{2n}\right)[\eta(X)\{g(U,Z)\eta(Y)\\&-g(Y,U)\eta(Z)\}-g(X,Y)\{g(Z,U)+\eta(U)\eta(Z)\}\\&-\eta(Y)\{g(Z,U)\eta(X)-g(X,U)\eta(Z)\}\\&+g(X,Z)\{g(Y,U)+\eta(Y)\eta(U)\}\\&-\eta(Z)\{g(X,U)\eta(Y)-g(Y,U)\eta(X)\}\\&-\eta(U)\{g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\}]=0.\end{aligned} \quad (3.15)$$

Let $\{e_i : i=1,2,...,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=Y=e_i$ in (3.15) and taking summation over $i, 1\leq i\leq 2n+1$, we get

$$S(Z,U) = \frac{r + 2n(2n+\lambda)}{2n+1} g(Z,U). \quad (3.16)$$

Again taking an orthonormal frame field at any point of the manifold and contracting over Z and U in (3.16) we have $\lambda = -2n < 0,$ for $n > 1.$ Hence the Ricci soliton is shrinking. This completes the proof of the theorem.

Theorem 3.3: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold and (g, V, λ) be a Ricci soliton satisfying the condition $W_2(\xi, X).S = 0$ in M, then the Ricci soliton is steady.

Proof: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold and (g, V, λ) be a Ricci soliton in M. Suppose that the condition $W_2(\xi, X).S(Y,Z) = 0$ holds in M, then we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0. \quad (3.17)$$

In view of (3.10), (2.15) and (3.17) we obtain

$$\begin{aligned}
&\lambda\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\} - \frac{1}{2n}\{S(QX,Z)\eta(Y) \\
&+ S(QX,Y)\eta(Z)\} + S(X,Z)\eta(Y) + S(X,Y)\eta(Z) \\
&+ g(X,Y)S(Q\xi, Z) + g(X,Z)S(Q\xi, Y) = 0.
\end{aligned} \quad (3.18)$$

Putting $Z = \xi$ in (3.18) and using (2.1), (2.3) and (2.15) we get

$$\begin{aligned}
&S(QX,Y) \\
&= 2n[\lambda\{(\lambda+1)g(X,Y) + \frac{\lambda}{2n}\eta(X)\eta(Y)\} \\
&\quad + S(X,Y) - \eta(X)S(Q\xi, Y)].
\end{aligned} \quad (3.19)$$

Again taking $Y = \xi$ in (3.19) and using (2.1), (2.3) and (2.15) we obtain

$$(2n-1)\lambda^2\eta(X) = 0, \quad (3.20)$$

since $\eta(X) \neq 0,$ (3.20) implies that $\lambda = 0.$ Thus the Ricci soliton is steady. This proves the theorem.

Theorem 3.4: A Ricci soliton in a $(2n+1)$-dimensional LP-Sasakian manifold satisfying the condition $W_2(\xi, X).W_4 = 0$ is shrinking under the condition $tr.\varphi = 0.$

Proof: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold and (g, V, λ) be a Ricci soliton in M. The W_4-curvature tensor in M is defined by [15]

$$\begin{aligned}
&W_4(X,Y,Z,T) \\
&= R(X,Y,Z,T) + \frac{1}{2n}[g(X,Z)Ric(Y,T) \\
&\quad - g(X,Y)Ric(Z,T)]
\end{aligned}$$

which can be written as

$$\begin{aligned}
&W_4(X,Y)Z \\
&= R(X,Y)Z + \frac{1}{2n}[g(X,Z)QY \\
&\quad - g(X,Y)QZ].
\end{aligned} \quad (3.21)$$

Putting $X = \xi$ in (3.21) and using (2.3) and (2.8) we obtain

$$\begin{aligned}
W_4(\xi, Y)Z &= g(Y,Z)\xi - \eta(Z)Y \\
&+ \frac{1}{2n}[\eta(Z)QY - \eta(Y)QZ].
\end{aligned} \quad (3.22)$$

Taking inner product on both sides of (3.21) with ξ and using (2.7) and (2.15) we get

$$\begin{aligned}
&\eta(W_4(X,Y)Z) \\
&= g(Y,Z)\eta(X) + \frac{\lambda}{2n}g(X,Y)\eta(Z) \\
&\quad - \left(1 + \frac{\lambda}{2n}\right)g(X,Z)\eta(Y).
\end{aligned} \quad (3.23)$$

Now, we assume that the condition $W_2(\xi, X).W_4(Y,Z)U = 0$ holds in M, then we have

$$\begin{aligned}
&W_2(\xi, X)W_4(Y,Z)U \\
&- W_4(W_2(\xi, X)Y, Z)U \\
&- W_4(Y, W_2(\xi, X)Z)U \\
&- W_4(Y,Z)W_2(\xi, X)U = 0.
\end{aligned} \quad (3.24)$$

In view of (3.10) and (3.24) we get

$$\begin{aligned}
&g(W_4(Y,Z)U, X)\xi - \eta(W_4(Y,Z)U)X \\
&+ \frac{1}{2n}[\eta(W_4(Y,Z)U)QX \\
&- g(W_4(Y,Z)U, X)Q\xi - \eta(Y)W_4(QX, Z)U \\
&+ g(X,Y)W_4(Q\xi, Z)U - \eta(Z)W_4(Y, QX)U \\
&+ g(X,Z)W_4(Y, Q\xi)U - \eta(U)W_4(Y,Z)QX \\
&+ g(X,U)W_4(Y,Z)Q\xi] - g(X,Y)W_4(\xi, Z)U \\
&+ \eta(Y)W_4(X,Z)U - g(X,Z)W_4(Y,\xi)U \\
&+ \eta(Z)W_4(Y,X)U - g(X,U)W_4(Y,Z)\xi \\
&+ \eta(U)W_4(Y,Z)X = 0.
\end{aligned} \quad (3.25)$$

Taking inner product on both sides of (3.25) with ξ and using (2.1), (2.3) and (2.15) we obtain

$$\left(1+\frac{1}{2n}\right)\{g(W_4(Y,Z)U,X)+\eta(X)\eta(W_4(Y,Z)U)\}$$
$$+\frac{1}{2n}\{\eta(Y)\eta(W_4(QX,Z)U)-g(X,Y)\eta(W_4(Q\xi,Z)U)$$
$$+\eta(Z)\eta(W_4(Y,QX)U)-g(X,Z)\eta(W_4(Y,Q\xi)U)$$
$$+\eta(U)\eta(W_4(Y,Z)QX)-g(X,U)\eta(W_4(Y,Z)Q\xi)\}$$
$$+g(X,Y)\eta(W_4(\xi,Z)U)-\eta(Y)\eta(W_4(X,Z)U)$$
$$+g(X,Z)\eta(W_4(Y,\xi)U)-\eta(Z)\eta(W_4(Y,X)U)$$
$$+g(X,U)\eta(W_4(Y,Z)\xi)-\eta(U)\eta(W_4(Y,Z)X)=0. \quad (3.26)$$

In view of (3.21), (3.23), (3.26) and (2.15) we get

$$\left(1+\frac{\lambda}{2n}\right)[g(R(Y,Z)U,X)-g(X,Y)g(Z,U)$$
$$+\frac{1}{2n}\{g(Y,U)S(X,Z)-g(Y,Z)S(X,U)$$
$$+S(X,Z)\eta(U)\eta(Y)+\lambda g(X,Z)\eta(U)\eta(Y)$$
$$-\lambda g(Y,Z)g(X,U)\}+\left(1+\frac{\lambda}{2n}\right)g(X,Z)g(Y,U)]$$
$$-\frac{\lambda}{4n^2}\{S(X,U)+\lambda g(X,U)\}\eta(Y)\eta(Z)$$
$$-\frac{1}{2n}\{\lambda g(X,Y)+S(X,Y)\}\eta(U)\eta(Z)=0. \quad (3.27)$$

Let $\{e_i:1,2,...,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=Y=e_i$ in (3.27) and summing over $i, 1\le i\le 2n+1$, we get

$$S(U,Z)=2ng(U,Z)+\left\{\frac{(2n+1)\lambda+r}{2n+\lambda}\right\}\eta(U)\eta(Z). \quad (3.28)$$

Again taking $U=Z=\xi$ and using (2.1), (2.3), (2.14) and (2.15) we get $\lambda=-2n<0.$ Thus λ is negative. This concludes that the Ricci soliton is shrinking. This completes the proof of the theorem.

Theorem 3.5: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold and (g,V,λ) be a Ricci soliton in M. If g satisfies the condition $W_4(\xi,X).W_2=0,$ then g is shrinking under the condition $tr.\varphi=0.$

Proof: Let M be a $(2n+1)$-dimensional LP-Sasakian manifold and (g,V,λ) be a Ricci soliton in M. Suppose that the condition $W_4(\xi,X).W_2(Y,Z)U=0$ holds in M, then by definition we have

$$0=W_4(\xi,X)W_2(Y,Z)U-W_2(W_4(\xi,X)Y,Z)U$$
$$-W_2(Y,W_4(\xi,X)Z)U-W_2(Y,Z)W_4(\xi,X)U. \quad (3.29)$$

By virtue of (3.22) and (3.29) we have

$$g(W_2(Y,Z)U,X)\xi-\eta(W_2(Y,Z)U)X$$
$$+\frac{1}{2n}[\eta(W_2(Y,Z)U)QX-\eta(X)QW_2(Y,Z)U$$
$$-\eta(Y)W_2(QX,Z)U+\eta(X)W_2(QY,Z)U$$
$$-\eta(Z)W_2(Y,QX)U+\eta(X)W_2(Y,QZ)U$$
$$-\eta(U)W_2(Y,Z)QX+\eta(X)W_2(Y,Z)QU] \quad (3.30)$$
$$-g(X,Y)W_2(\xi,Z)U+\eta(Y)W_2(X,Z)U$$
$$-g(X,Z)W_2(Y,\xi)U+\eta(Z)W_2(Y,X)U$$
$$-g(X,U)W_2(Y,Z)\xi+\eta(U)W_2(Y,Z)X$$
$$=0.$$

Taking inner product on both sides of (3.30) with ξ and using (2.1), (2.3) and (2.15) we obtain

$$g(W_2(Y,Z)U,X)+\eta(X)\eta(W_2(Y,Z)U)$$
$$+\frac{1}{2n}[\lambda\eta(X)\eta(W_2(Y,Z)U)+\eta(X)g(QW_2(Y,Z)U,\xi)$$
$$+\eta(Y)\eta(W_2(QX,Z)U)-\eta(X)\eta(W_2(QY,Z)U)$$
$$+\eta(Z)\eta(W_2(Y,QX)U)-\eta(X)\eta(W_2(Y,QZ)U)$$
$$+\eta(U)\eta(W_2(Y,Z)QX)-\eta(X)\eta(W_2(Y,Z)QU)]$$
$$+g(X,Y)\eta(W_2(\xi,Z)U)-\eta(Y)\eta(W_2(X,Z)U)$$
$$+g(X,Z)\eta(W_2(Y,\xi)U)-\eta(Z)\eta(W_2(Y,X)U)$$
$$+g(X,U)\eta(W_2(Y,Z)\xi)-\eta(U)\eta(W_2(Y,Z)X)=0. \quad (3.31)$$

In view of (3.9) and (3.11), (3.31) yields

$$g(R(Y,Z)U,X)+\frac{1}{2n}\{g(Y,U)S(X,Z)$$
$$-g(Z,U)S(X,Y)\}+\left(1+\frac{\lambda}{2n}\right)[g(Y,U)g(X,Z)$$
$$-g(X,Y)g(Z,U)+\frac{1}{n}\{S(Y,U)\eta(X)\eta(Z)$$
$$-S(Z,U)\eta(X)\eta(Y)+\frac{1}{2}S(X,Z)\eta(Y)\eta(U)$$
$$-\frac{1}{2}S(X,Y)\eta(U)\eta(Z)\}]=0. \quad (3.32)$$

Let $\{e_i:i=1,2,...,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=Y=e_i$ in (3.32) and summing over $i, 1\le i\le 2n+1,$ we get

$$S(Z,U)=\left(\frac{8n^3+4n^2\lambda+2nr}{4n^2+6n+2\lambda}\right)g(U,Z)+\left(\frac{6n\lambda+2nr+3\lambda^2+\lambda r}{4n^2+6n+2\lambda}\right)\eta(U)\eta(Z). \quad (3.33)$$

Again, putting $Z=U=\xi$ in (3.33) and using (2.1), (2.3) and (2.15) we get

$$\lambda^2+4n\lambda+4n^2=0. \quad (3.34)$$

This equation gives $\lambda = -2n, -2n$. Hence λ is negative. This concludes that the Ricci soliton is shrinking. Thus the theorem is proved.

From theorem 3.4 and theorem 3.5 we can state next theorem

Theorem 3.6: Ricci solitons in a $(2n+1)$-dimensional LP-Sasakian manifold satisfying the derivation conditions $W_2(\xi, X).W_4 = 0$ and $W_4(\xi, X).W_2 = 0$ are equivalent.

Now we give an example of LP-Sasakian manifold.

4. Example for 3-dimensional LP-Sasakian Manifold

Let us consider a 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in R^3\}$, where (x, y, z) are standard coordinates in R^3. We choose the vector fields

$$E_1 = -e^x \frac{\partial}{\partial y}, E_2 = e^x\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right), E_3 = \frac{\partial}{\partial x}, \quad (4.1$$

which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0, \qquad (4.2)$$

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = -1. \quad (4.3)$$

Let η be a 1-form defined by $\eta(Z) = g(Z, E_3)$ for any vector field Z on M. Let φ be a (1, 1) tensor field defined by

$$\varphi(E_1) = -E_1, \varphi(E_2) = -E_2, \varphi(E_3) = 0. \qquad (4.4)$$

The linearity property of φ and g yields that

$$\eta(E_3) = -1, \varphi^2(U) = U + \eta(U)E_3, \qquad (4.5)$$

$$g(\varphi Z, \varphi U) = g(Z, U) + \eta(Z)\eta(U), \qquad (4.6)$$

for any vector fields Z, U on M. Thus for $E_3 = \xi, (\varphi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on M.

By the definition of Lie bracket and (4.1) we have

$$[E_1, E_3] = -E_1, [E_1, E_2] = 0, [E_2, E_3] = -E_2. \quad (4.7)$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g, the Koszul formula is defined as

$$\begin{aligned} &2g(\nabla_X Y, Z) \\ &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) \\ &\quad + g(Z, [X, Y]). \end{aligned} \qquad (4.8)$$

In view of (4.2), (4.3), (4.7) and (4.8) we get

$$\begin{aligned} &2g(\nabla_{E_1} E_3, E_1) \\ &= E_1 g(E_3, E_1) + E_3 g(E_1, E_1) - E_1 g(E_1, E_3) \\ &\quad - g(E_1, [E_3, E_1]) - g(E_3, [E_1, E_1]) + g(E_1, [E_1, E_3]) \\ &= -2g(E_1, E_1). \end{aligned}$$

Similarly, we can obtain

$$2g(\nabla_{E_1} E_3, E_2) = 0 = -2g(E_1, E_2)$$

and $2g(\nabla_{E_1} E_3, E_3) = 0 = -2g(E_1, E_3)$.

From above we can write $2g(\nabla_{E_1} E_3, X) = -2g(E_1, X)$ for all $X \in \chi(M)$. Thus $\nabla_{E_1} E_3 = -E_1$.

Proceeding same way we obtain

$$\begin{cases} \nabla_{E_1} E_3 = -E_1, \nabla_{E_1} E_2 = 0, \\ \nabla_{E_1} E_1 = -E_3, \nabla_{E_2} E_3 = -E_2, \\ \nabla_{E_2} E_2 = -E_3, \nabla_{E_2} E_1 = 0, \\ \nabla_{E_3} E_3 = 0 = \nabla_{E_3} E_2 = \nabla_{E_3} E_1. \end{cases} \qquad (4.9)$$

Now, we have

$$\begin{aligned} (\nabla_{E_1} \varphi) E_1 &= \nabla_{E_1} \varphi E_1 - \varphi \nabla_{E_1} E_1 \\ &= -\nabla_{E_1} E_1 + \varphi(E_3) \\ &= E_3. \end{aligned}$$

Again, from definition and by the use of (2.5) we obtain

$$\begin{aligned} &(\nabla_{E_1} \varphi) E_1 \\ &= g(E_1, E_1) E_3 + \eta(E_1) E_1 + 2\eta(E_1)\eta(E_1) E_3 \\ &= E_3. \end{aligned}$$

Similarly, we obtain other relations. Thus we have

$$\begin{cases} (\nabla_{E_1} \varphi) E_1 = E_3, (\nabla_{E_1} \varphi) E_2 = 0, \\ (\nabla_{E_1} \varphi) E_3 = -E_1, (\nabla_{E_2} \varphi) E_1 = 0, \\ (\nabla_{E_2} \varphi) E_2 = E_3, (\nabla_{E_2} \varphi) E_3 = -E_2, \\ (\nabla_{E_3} \varphi) E_1 = 0, (\nabla_{E_3} \varphi) E_2 = 0, \\ (\nabla_{E_3} \varphi) E_3 = 0. \end{cases} \qquad (4.10)$$

From (4.5), (4.6), (4.9) and (4.10), we see that the equations (2.1) - (2.5) are satisfied by the manifold M, for $E_3 = \xi$. Hence (φ, ξ, η, g) is an LP-Sasakian structure in M. Consequently $M^3(\varphi, \xi, \eta, g)$ is an LP-Sasakian manifold.

Now, the Riemannian curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad (4.11)$$

By virtue of (4.7), (4.9) and (4.11) we obtain

$$R(E_1,E_2)E_3 = \nabla_{E_1}\nabla_{E_2}E_3 - \nabla_{E_2}\nabla_{E_1}E_3 - \nabla_{[E_1,E_2]}E_3 = -\nabla_{E_1}E_2 + \nabla_{E_2}E_1 = 0.$$

Similarly, we obtain

$$\begin{cases} R(E_1,E_2)E_3 = 0, R(E_2,E_3)E_3 = -E_2, \\ R(E_1,E_3)E_3 = -E_1, R(E_1,E_2)E_2 = E_1, \\ R(E_2,E_3)E_2 = -E_3, R(E_1,E_3)E_2 = 0, \\ R(E_1,E_2)E_1 = -E_2, R(E_2,E_3)E_1 = 0, \\ R(E_1,E_3)E_1 = -E_3, R(E_1,E_1)E_1 = 0 \\ R(E_2,E_2)E_2 = R(E_3,E_3)E_3 = 0. \end{cases} \quad (4.12)$$

By the use of (4.12) we get

$$S(E_1,E_1) = \sum_{i=1}^{3} g(R(E_1,E_i)E_i,E_1) = g(R(E_1,E_2)E_2,E_1) + g(R(E_1,E_3)E_3,E_1) = g(E_1,E_1) + g(-E_1,E_1) = 0.$$

Similarly, we obtain $S(E_2,E_2)=0$ and $S(E_3,E_3)=-2.$ Thus we have

$$\begin{cases} S(E_1,E_1) = S(E_2,E_2) = 0, \\ S(E_3,E_3) = -2. \end{cases} \quad (4.13)$$

From (2.13) we have

$$S(E_i,E_i) = -\{\lambda g(E_i,E_i) + g(E_i,\varphi E_i)\}.$$

This equation yields

$$S(E_1,E_1) = S(E_2,E_2) = -(\lambda-1),$$

by the use of (4.3), (4.4) and (4.13) for $i = 1, 2.$ This implies $\lambda = 1 > 0$, for $i = 1,2.$ And

$$S(E_3,E_3) = \lambda, \quad \text{for } i = 3.$$

This yields $\lambda = -2 < 0.$ Since $\lambda = 1 > 0$ for $i = 1,2$ and $\lambda = -2 < 0$ for $i = 3$, this is an example of expanding and shrinking Ricci soliton in 3-dimensional LP-Sasakian manifold.

5. Conclusions

In this paper, we have investigated that the Ricci soliton in a $(2n+1)$-dimensinal LP-Sasakian manifold is shrinking. It is also proved that Ricci solitons in an LP-Sasakian manifold satisfying the derivation conditions $R(\xi,X).W_2 = 0$, $W_2(\xi,X).W_4 = 0$ and $W_4(\xi,X).W_2 = 0$ are shrinking but are steady for the condition $W_2(\xi,X).S = 0.$

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