PROOF IN MATHEMATICS

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INTRODUCTION

The aim is to set our the general framework in which a mathematical proof is possible, and then to examine the various kinds of proof which are common to all branches of mathematics.

In the study of mathematics it is not possible to avoid proof, and once we are faced with the problem of providing a mathematical conjecture we are in a situation where mathematics and logic are both involved.

There are mainly two methods to solve the mathematical problems. They are methods of direct proof and methods of indirect proof.

METHODS OF DIRECT PROOF

A direct proof to be a chain of argument which leads directly from axioms and definition to the theorem which we wish to prove.

Example – 1

Given : The set $\{a,b,c,d\}$

The binary operation "$\circ$" on the set defined by

$\circ$	a	b	c	d
a	a	c	d	b
b	c	b	a	d
c	d	a	c	a
d	b	d	a	d

(Note that : In the tables the combination $c \circ d$, say, is found as the element, a, in the intersection of the row beginning with c at the left end and the column beginning with d at the top.)

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To prove "$\circ$" is commutative.

From the table, we have (excluding $a \circ a$, $b \circ b$, etc.)

$$a \circ b = b \circ a$$

$$a \circ c = c \circ a$$

$$a \circ d = d \circ a$$

$$b \circ c = c \circ b$$

$$b \circ d = d \circ b$$

$$c \circ d = d \circ c$$

Hence, "$\circ$" is commutative.

Example – 2

To prove

$$\left.\begin{array}{l} a+b=-p \\ ab=q \end{array}\right\} \quad (a,b \in R)$$

are necessary and sufficient conditions for a, b to be the roots of the equation. x^2+px+q, where $p^2>4q$.

Proof :

Let $\mathbf{a}$ represents

$$a+b=-p \text{ and } ab=q \qquad (a,b \in R)$$

and $\mathbf{b}$ represents

$$a,b \text{ are roots of the equation } x^2+px+q=0$$

We need to show

$\mathbf{a} \Rightarrow \mathbf{b}$ (sufficient)

and $\mathbf{b} \Rightarrow \mathbf{a}$ (necessity)

SUFFICIENT

Assume that $\mathbf{a}$ is true; that is assume

$$a+b=-p$$

and $ab=q$.

Then,

$$b=-p-a$$

$$\therefore \quad ab = a(-p-a) = -pa - a^2$$

Also,

$$ab = q$$

$$\therefore \quad a^2 + pa + q = 0$$

$\therefore$ a is a root pf the given equation.

Similarly,

b is a root of the given equation.

Thus b is the true whenever a is true, that is $\mathbf{a} \Rightarrow \mathbf{b}$.

NECESSARY

Assume that b is true; that is assume a, b to be roots of the given equation.

By the formula for the roots of a quadratic, we have (say)

$$a = \frac{-p+\sqrt{p^2-4q}}{2}$$

$$b = \frac{-p-\sqrt{p^2-4q}}{2}$$

$$\therefore \quad a + b = -p$$

and $ab = \frac{(-p)^2 - (p^2 - 4q)}{4} = q$

Thus a is true whenever b is true, that is, $\mathbf{b} \Rightarrow \mathbf{a}$. So $\mathbf{a} \Rightarrow \mathbf{b}$ and $\mathbf{b} \Rightarrow \mathbf{a}$, and the proof is complete.

METHODS OF INDIRECT PROOF

Indirect proof as a proof of a proposition equivalent to that which we want to prove, and we pointed out that in general the rule of substitution allows us to turn an indirect proof into direct proof by adding one further step.

However, there is one particular method of indirect proof, in which we prove that ~ a is *FALSE* when we want to prove that a is *TRUE*, and we prove ~ a is *FALSE* by assuming that it is *TRUE* and showing that this leads to a contradiction. Such a proof is called proof by contradiction. (reductio and absurdum).

This method of proof is used very frequently in mathematics, and there are a number of theorem for which no other method of proof has been discovered.

Example – 1

To prove

For a square, the ratio : $\frac{\text{length of side}}{\text{lenth of diagonal}}$ cannot be expressed as $\frac{x}{y}$,

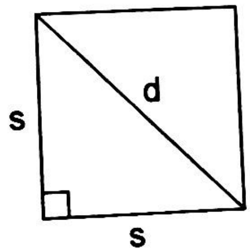
when $x, y \in Z^{+}$ and have no common divisor.

PROOF BY CONTRADICTION

Let *s* be the length of a side and let *d* be the length of a diagonal of a square. Then assume as a hypothesis the contradiction of the conjecture; that is, if the hypothesis is a, we assume that ~ a is *TRUE*, that is

$\frac{s}{d} = \frac{x}{y}$ (where x and y have no common divisor)

$\therefore \quad \frac{s^2}{d^2} = \frac{x^2}{y^2}$



By Pythagoras' theorem, $d^2 = 2s^2$, so

$$\frac{s^2}{d^2} = \frac{s^2}{2s^2} = \frac{1}{2}$$

But $\frac{s^2}{d^2} = \frac{x^2}{y^2}$

$\therefore \quad \frac{x^2}{y^2} = \frac{1}{2}.$

Now

$y^2 = 2x^2 \Rightarrow y^2$s even $\Rightarrow$ y is even.

(since the square of an odd number is odd), and x is odd, since x, y have no common divisor. Further, if y is even, then

$y = 2z$ for some integer z.

$\therefore \quad y^2 = 4z^2$

But $\quad y^2 = 2x^2$

$\therefore \quad 2x^2 = 4z^2$

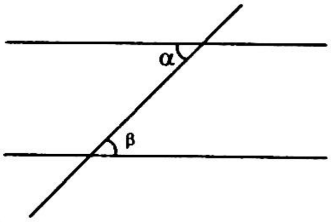
and

$x^2 = 2z^2 \Rightarrow x^2$ is even $\Rightarrow$ x is even. We have thus proved that x is odd and also that x is even, which is a contradiction.

Since the conclusion is contradictory and reasoning vaid, the hypothesis $\sim a$ is must be *FALSE*. If the hypothesis is *FALSE*, the original conjecture a must be *TRUE*.

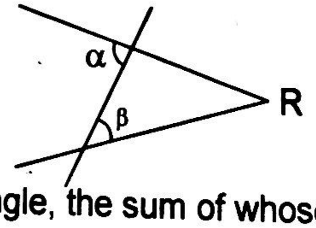
Example – 2

Prove that, in Euclidean geometry, if straight line cuts two other straight lines in a plane so that the alternate angles are equal, then the two cut straight lines do not interest.



By using contradiction and use the theorem that the sum of the interior angles of a triangle is 180^0.

Assume that the cut lines interest at R. let the acute angle between the lines at R be γ.



Then we have a triangle, the sum of whose interior angle is

$$\gamma + \beta + (180^0 - \alpha) = \gamma + \beta + (180^0 - \beta)$$

$$= \gamma + 180^0$$

But the sum of the interior angles of every triangle is 180^0. this is a contradiction, since $\gamma > 0$.

CONCLUSION

In direct proof, it proceed by a series of steps, each step using a rule of inference, from what is given or assumed to what is to be proved.

An alternative type of proof, known as indirect proof proves a proposition which is equivalent to what is to be proved. We can easily see, therefore, that the rule of equivalence enables us to convert an indirect proof to a direct proof by the addition of one further step in the chain of reasoning. The most common form of indirect proof in mathematics involves proving that a proposition, contradictory to what we want to prove.

REFERENCES

1 Mathematics foundation course unit 17 LOGIC II – PROOF. The Open University press (Walton Hall, Bletchley, Bucks)

2 Mittal P.K. and Gupta K.C. An integral involving generalized function of two variables. India. Acad. Sci.

3. Mathematical analysis – II Sharma-Vasishtha (K. Prakashan Meerut, India))