Maxwell's equations and their significance

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Introduction

In electromagnetism, Maxwell's equations are a set of four partial differential equations that describe the properties of the electric and magnetic fields and relate them to their sources, charge density and current density. Individually, the equations are known as Gauss's law in electrostatic, Gauss's law in magnetism, Faraday's law of electromagnetic induction, and Ampere's law with Maxwell's correction. These four equations, together with Lorentz force law are the complete set of laws of classical electromagnetism.

Maxwell's contribution to Science in producing these equations lies in the correction he made to Ampere's circuital law in his 1861 paper *'On physical lines of force'*. In an attempt to account the continuity of current flow in the space between the plates of a capacitor, James Clerk Maxwell (1831-1879), the great Physicist, in 1861 introduced the concept of displacement current. The idea of displacement current brought together the phenomena of electricity and magnetism into a unified theory. The introduction of displacement current led Maxwell to modify Ampere's circuital law

$$\oint \vec{B}.d\vec{l} = \mu_0 I$$

to the form

$$\oint \vec{B}.d\vec{l} = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

stating that a time varying electric field set up in vacuum or in a dielectric induces a magnetic field, as a current flowing through conductor does. Maxwell then summarized the fundamental known laws of electricity and magnetism, namely Gauss's laws, Ampere's law and Faraday's law of electromagnetism in new and precise, and unified ways in terms of electric and magnetic fields and their sources, which in honour of him , are called Maxwell's Equations. Maxwell's equations were published by Maxwell in 1873 as "General Equations of Electromagnetic Field" in volume 2 of his "A Treastise on Electricity and Magnetism." This book by Maxwell predates publications by Heaviside, Hertz and others.

Maxwell also developed Faraday's law of induction into another equation which was used to be listed as a

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'Maxwell's equation' but is now-a-days known as the Lorentz force law.

Maxwell's equations which are the mathematical abstracts of experimental results, describe the behaviour of electromagnetic fields and form the foundation of electromagnetism.

The following table displays Maxwell's equations which apply to electric and magnetic fields in vacuum; they may also be generalized to include fields in matter.

Table: **Maxwell's Equations**

a) **Integral form**	b) **Differential form**	**Name**
i) $\oint_S \vec{E}.d\vec{S} = \dfrac{q}{\varepsilon_0}$	$\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss' law in electrostatics
ii) $\oint \vec{B}.d\vec{S} = 0$	$\vec{\nabla}.\vec{B} = 0$	Gauss' law in magnetostatics
iii) $\oint \vec{E}.d\vec{l} = -\dfrac{d\phi_B}{dt}$ $= -\dfrac{d}{dt}\oint \vec{B}.d\vec{S}$	$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	Faraday's law of e.m. induction
iv) $\oint \vec{B}.d\vec{l} = \mu_0\left(I + \varepsilon_0\dfrac{d\phi_E}{dt}\right)$ $= \mu_0\left(I + \varepsilon_0\dfrac{d}{dt}\oint\vec{E}.d\vec{S}\right)$ or $\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon_0\dfrac{\partial\vec{E}}{dt}$	$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \varepsilon_0\dfrac{\partial\vec{E}}{dt})$	Ampere –Maxwell law

Significance of Maxwell's equations

Maxwell's synthesis of electromagnetism in these four equations is one of the great milestones of theoretical physics, in compared with Newton's laws of motion in mechanics. The physical significance of these equations is that each of them represents a generalization of certain experimental observations and results, which can be summarized in the following points:

i) The first equation $\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$ relates electric charge contained within a closed surface to the surrounding

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electric field. It describes with mathematical clarity how the divergence of an electric field is affected by charges. It states that net outward flux of electric displacement vector through closed surface is equal to the net charge enclosed by the surface but is unrelated to the shape and size of that surface.

ii) The second equation $\vec{\nabla}.\vec{B} = 0$ states that total magnetic flux through any closed surface is zero. This is due to real world magnetic charges coming in pairs(referred to as magnetic dipoles), with the two charges giving rise to opposite magnetic field divergences which cancel each other out . Gauss's law for magnetism is also mathematical form of the assertion that single magnetic charges, referred to as magnetic monopoles, do not exists in our physical world.

iii) The third equation $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

describes how a time varying magnetic field can create an electric field and acts as the source of induced e.m.f. This is the operating principle behind many electric generators like hydroelectric generator that drives electricity through the power grid .

iv) The fourth equation $\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \dfrac{\partial \vec{E}}{dt})$, Ampere's law with Maxwell's correction, states that magnetic fields can be regenerated in two ways: by electric current and by changing electric fields. The idea that a magnetic field can be induced by a changing electric field follows from the modern concept of displacement current which was introduced to maintain the solenoidal nature of

Ampere's law in vacuum capacitor circuit. Maxwell's current applies to polarization current in a dielectric medium, and it sits adjacent to the modern displacement current in Ampere's law.

Consequences of Maxwell's equations

Maxwell's equations are not only the fundamental equations of electromagnetism, but are also foundations of many of the equations of other fields of physics such as Optics etc. They are the basis for the functioning of such electromagnetic devices as electric motors, cyclotrons, radio and television transmitters and receivers, telephones, fax machines, radar and microwave ovens etc. These equations have several consequences and explain a diverse range of physical phenomena.

Some important consequences of the equations can be listed as follows:

i) Electric charge is conserved, according to the equation of continuity

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 .$$

ii) In 1864, Maxwell developed the electromagnetic wave equation by linking the displacement current to the changing electric field that is associated with electromagnetic induction. One set of solutions to Maxwell's equations is in the form of travelling sinusoidal plane waves with the electric and magnetic field directions orthogonal to one another and the direction of travel, and with the two fields in phase when travelling

in free space or in isotropic dielectric medium but in different phases when travelling in conducting medium. In fact Maxwell's equations explain specifically how these waves can physically propagate through space. The changing magnetic field creates changing electric field through Faraday's law. That electric field, in turn, creates a changing magnetic field through Maxwell's correction to Ampere's law. This perpetual cycle allows these waves, known as electromagnetic radiations, to move through space, always at velocity

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

Maxwell discovered that this quantity c equals the speed of light in vacuum (known from early experiments) and concluded that light is a form of electromagnetic radiation. The speed of electromagnetic waves in a medium is a function of electric and magnetic properties of the medium ($v = \frac{1}{\sqrt{\mu \varepsilon}}$) and is independent of the magnitude of vectors $\vec{E}$ and $\vec{B}$ (or $\vec{H}$). Speed of the electromagnetic wave is less in a medium than in vacuum ($v < c$).

The amplitudes of electric and magnetic field vectors in free space are related as $\vec{E} = c\,\vec{B}$.

iii) The fields $\vec{E}$ and $\vec{B}$ are derivable from potential functions:

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

Here $\vec{A}$ is vector potential and ϕ is scalar potential.

The potentials satisfy the inhomogeneous wave equations

$$\nabla^2 \phi - \varepsilon\mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\varepsilon}\rho \qquad \text{and}$$

$$\nabla^2 \vec{A} - \varepsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

if the Lorentz condition $\vec{\nabla}.\phi = -\varepsilon\mu \frac{\partial \phi}{\partial t}$ is imposed.

iv) Maxwell's Equations predict that energy in the electromagnetic waves is equally divided between oscillating electric and magnetic fields

($\vec{E}.\vec{D} = \vec{B}.\vec{H}$) and total energy is conserved, according to

$$\vec{\nabla}.\vec{S} + \frac{\partial u}{\partial t} = -\vec{J}.\vec{E},$$

where the field energy density, in a linear medium, is

$$u = \frac{1}{2}\left(\vec{E}.\vec{D} + \vec{B}.\vec{H} \right),$$

and the energy flux per unit area is the Poynting vector

$$\vec{S} = \vec{E} \times \vec{H}.$$

v) Electromagnetic waves are known to possess momentum p, equal to $1/c$ times the energy of the wave ($p = E/c$) and to exert radiation pressure P_r , equal to $1/c$ times the intensity of the wave ($P_r = I/c$). An

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evidence of radiation pressure is that of comets which form tails that point away from the sun.

vi) Maxwell's equations have a close relation to special relativity as they were a crucial part of the historical development of special relativity. Under Lorentz transformations, Maxwell's Equations are invariant in all inertial frames of reference in contrast to the famous Newtonian equations for classical mechanics.

General relativity has also had a close relationship with Maxwell's Equations. In general relativity matter and energy generate curvature of space-time. Curvature of space-time affects electrodynamics. An electromagnetic field having energy and momentum will also generate curvature in space-time. Maxwell's equations in curved space-time can be obtained by replacing the derivatives in the equations in flat space-time with covariant derivatives. Kaluja and Klein showed in the 1920s that Maxwell's equations can be derived by extending general relativity into five dimensions. This strategy of using higher dimensions to unify different forces continues to be an active area of research in Particle Physics.

With formulation of Maxwell's equations, Maxwell himself, in 1869, predicted theoretically the existence of electromagnetic waves but could not give experimental proof. In 1888, about 20 years later, it was confirmed experimentally by Heinrich Hertz. The use of electromagnetic waves for purpose of long distance communication, however, remained for enthusiasm and energy of Marconi to establish the first 'Wireless communication' across the English Channel, a distance of about 50 km.

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