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On contact conformal curvature tensor in LP-Sasakian manifolds

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Abstract

The purpose of the present paper is to study the contact conformal curvature tensor in LP-Sasakian manifolds. Some properties of contact conformally flat, ξ-contact conformally flat and contact conformally semi-symmetric LP-Sasakian manifolds are obtained.

Keywords: Contact conformal curvature tensor; LP-Sasakian manifold; η-Einstein manifold.

1. Introduction

The contact conformal curvature tensor is a curvature like tensor defined on a contact metric manifold which is constructed from the conformal curvature tensor by using the Boothby-Wang's fibration [1]. Jeong, Lee and Pak [2] defined the contact conformal curvature tensor on $(2n+1)$-dimensional Sasakian manifolds and proved that it is invariant under D-homothetic deformation. They also proved that a Sasakian manifold $M^{2n+1}(n>2)$ with vanishing contact conformal curvature tensor field is of constant φ-homothetic sectional curvature $[r-n(3n+1)]/n(n+1)$. Pak and Shin [3] gave a geometric characterization of a contact metric manifold with vanishing contact conformal curvature tensor by showing that for $n>2$, every $(2n+1)$-dimensional contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form. Bang and Kye [4] studied contact conformal curvature tensor on 3-dimensional Sasakian manifolds and gave a partial extension of Pak and Shin's result to 3-dimensional locally φ-symmetric contact metric manifold and also showed that the contact conformal curvature tensor on 3-dimensional Sasakian manifold always vanishes. On the other hand, Matsumoto [5] introduced the notion of Lorentzian para-Sasakian manifold. Then Mihai and Rosca [6] introduced the same notion independently and obtained many results on this manifold. Lorentzian para-Sasakian manifolds have also been studied by Matsumoto and Mihai [7], De *et al.* [8], Shaikh and Biswas [9] and Bagewadi *et al.* [10].

2. Preliminaries

A differentiable manifold of dimension $(2n+1)$ is called Lorentzian para-Sasakian manifold (briefly, LP-Sasakian manifold) if it admits a (1, 1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

(2.1) $$\eta(\xi)=-1, \varphi^2(X)=X+\eta(X)\xi,$$

(2.2) $$g(\varphi X,\varphi Y)=g(X,Y)+\eta(X)\eta(Y),$$

(2.3) $$g(X,\xi)=\eta(X),$$

(2.4) $$\nabla_X \xi=\varphi X,$$

(2.5) $$(\nabla_X \varphi)Y=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi,$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g [5,7].
In an LP-Sasakian manifold it can be easily seen that;

(2.6) $$\varphi\xi=0, \eta\circ\varphi=0, \operatorname{rank}\varphi=2n.$$ If we put

(2.7) $$\Phi(X,Y)=g(X,\varphi Y),$$

for any vector fields X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric (0, 2) tensor field [5]. Also since the 1-form η is closed in an LP-Sasakian manifold we have

(2.8) $$(\nabla_X \eta)(Y)=\Phi(X,Y)=g(X,\varphi Y)=g(\varphi X,Y), \Phi(X,\xi)=0,$$

for any vector fields X and Y [5,9]. An LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S of type (0, 2) is of the form

(2.9) $$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

for any vector fields X and Y, where a,b are smooth functions on the manifold. In particular, if $b=0$, then the manifold is said to be an Einstein manifold. In a $(2n+1)$-dimensional LP-Sasakian manifold the following relations hold:

(2.10) $$\eta(R(X,Y)Z)=g(Y,Z)\eta(X)-g(X,Z)\eta(Y),$$

(2.11) $$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X,$$

(2.12) $$R(X,Y)\xi=\eta(Y)X-\eta(X)Y,$$

(2.13) $$R(\xi,X)\xi=X+\eta(X)\xi,$$

(2.14) $$S(X,\xi)=2n\eta(X),$$

(2.15) $$S(\varphi X,\varphi Y)=S(X,Y)+2n\eta(X)\eta(Y),$$

for any vector fields X, Y and Z, where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold, respectively [8, 9]. In a $(2n+1)$-dimensional LP-Sasakian manifold the contact conformal curvature tensor C_0 of type (1, 3) is defined by [2] can be written as

(2.16)
$$\begin{aligned}
C_0(X,Y)Z = R(X,Y)Z &+ \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\
&- g(X,Z)QY + \eta(Y)S(X,Z)\xi - \eta(X)S(Y,Z)\xi + \eta(X)\eta(Z)QY \\
&- \eta(Y)\eta(Z)QX + S(\varphi X,Z)\varphi Y - S(\varphi Y,Z)\varphi X + g(\varphi X,Z)Q(\varphi Y) \\
&- g(\varphi Y,Z)Q(\varphi X) + 2g(\varphi X,Y)Q(\varphi Z) + 2S(\varphi X,Y)\varphi Z\} \\
&+ \frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)r}{2n}\}\{g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y \\
&- 2g(\varphi X,Y)\varphi Z\} + \frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)r}{2n}\}\{g(Y,Z)X \\
&- g(X,Z)Y\} - \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)r}{2n}\}\{\eta(Y)\eta(Z)X \\
&- \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\},
\end{aligned}$$

where R, S, Q and r denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.

Definition 2.1 A $(2n+1)$-dimensional LP-Sasakian manifold M is said to be contact conformally flat if the condition

(2.17) $$C_0(X,Y)Z = 0$$

holds.

Definition 2.2 A $(2n+1)$-dimensional LP-Sasakian manifold M is said to be ξ-contact conformally flat if

(2.18) $$C_0(X,Y)\xi = 0.$$

Definition 2.3 A Riemannian or pseudo-Riemannian manifold is said to be semi-symmetric if the condition

(2.19) $$R(X,Y).R = 0$$

holds, where $R(X,Y)$ denotes the derivation in the tensor algebra at each point of the manifold.

Definition 2.4 A $(2n+1)$-dimensional LP-Sasakian manifold M is said to be contact conformally semi-symmetric if

(2.20) $$R(X,Y).C_0 = 0.$$

3. Results and Discussion

We prove the following results which are related with above definitions

Theorem 3.1 A contact conformally flat LP-Sasakian manifold M of dimension $(2n+1)$ is an η-Einstein manifold.

Proof: Let us consider a contact conformally flat LP-Sasakian manifold M, then (2.17) holds and from (2.16) we have

(3.1)
$$\begin{aligned}
0 = R(X,Y)Z + \frac{1}{2n}\{&S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\
&- g(X,Z)QY + \eta(Y)S(X,Z)\xi - \eta(X)S(Y,Z)\xi + \eta(X)\eta(Z)QY \\
&- \eta(Y)\eta(Z)QX + S(\varphi X,Z)\varphi Y - S(\varphi Y,Z)\varphi X + g(\varphi X,Z)Q(\varphi Y) \\
&- g(\varphi Y,Z)Q(\varphi X) + 2g(\varphi X,Y)Q(\varphi Z) + 2S(\varphi X,Y)\varphi Z\} \\
&+ \frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)r}{2n}\}\{g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y \\
&- 2g(\varphi X,Y)\varphi Z\} + \frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)r}{2n}\}\{g(Y,Z)X \\
&- g(X,Z)Y\} - \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)r}{2n}\}\{\eta(Y)\eta(Z)X \\
&- \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\}.
\end{aligned}$$

Taking inner product on both sides of (3.1) by W, we get

(3.2)

$$0=\tilde{R}(X,Y,Z,W)+\frac{1}{2n}\{S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)$$
$$-g(X,Z)S(Y,W)+\eta(Y)S(X,Z)\eta(W)-\eta(X)S(Y,Z)\eta(W)+\eta(X)\eta(Z)S(Y,W)$$
$$-\eta(Y)\eta(Z)S(X,W)+S(\varphi X,Z)g(\varphi Y,W)-S(\varphi Y,Z)g(\varphi X,W)+g(\varphi X,Z)S(\varphi Y,W)$$
$$-g(\varphi Y,Z)S(\varphi X,W)+2g(\varphi X,Y)S(\varphi Z,W)+2S(\varphi X,Y)g(\varphi Z,W)\}$$
$$+\frac{1}{2n(n+1)}\{2n^2-n-2+\frac{(n+2)r}{2n}\}\{g(\varphi Y,Z)g(\varphi X,W)-g(\varphi X,Z)g(\varphi Y,W)$$

$$-2g(\varphi X,Y)g(\varphi Z,W)\}+\frac{1}{2n(n+1)}\{n+2-\frac{(3n+2)r}{2n}\}\{g(Y,Z)g(X,W)$$
$$-g(X,Z)g(Y,W)\}-\frac{1}{2n(n+1)}\{4n^2+5n+2-\frac{(3n+2)r}{2n}\}\{\eta(Y)\eta(Z)g(X,W)$$
$$-\eta(X)\eta(Z)g(Y,W)+\eta(X)g(Y,Z)\eta(W)-\eta(Y)g(X,Z)\eta(W)\},$$

where $\tilde{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$. Setting $W=\xi$ in (3.2) and using (2.1), (2.3), (2.6), (2.10), (2.14) and then further simplifying yields

(3.3)
$$0=\left\{\frac{8n^3+10n^2+4n-(3n+2r)}{2n(2n+1)}\right\}\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}$$
$$+S(Y,Z)\eta(X)-S(X,Z)\eta(Y).$$

In (3.3) replacing X by ξ and using (2.1), (2.3) and (2.14), we get

(3.4)
$$S(Y,Z)=\left\{-\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}\right\}g(Y,Z)$$
$$+\left\{-\frac{2n(6n^2+7n+2)-(3n+2)r}{2n(n+1)}\right\}\eta(Y)\eta(Z).$$

Equation (3.4) implies that

(3.5)
$$S(Y,Z)=\alpha g(Y,Z)+\beta\eta(Y)\eta(Z),$$

where $\alpha=-\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}$ and $\beta=-\frac{2n(6n^2+7n+2)-(3n+2)r}{2n(n+1)}$. The relation (3.5) implies that the manifold is an η-Einstein manifold. This completes the proof of the theorem.

Theorem 3.2 Let M be a $(2n+1)$-dimensional LP-Sasakian manifold. If the condition $C_0(X,Y)\xi=0$ holds in M, then the manifold is an η-Einstein manifold.

Proof: Let us consider a $(2n+1)$-dimensional LP-Sasakian manifold M which is ξ-contact conformally flat, then we have $C_0(X,Y)\xi=0$. Now, replacing Z by ξ in (2.16) and using (2.1), (2.3), (2.6), (2.12), (2.14) and (2.18), we get

(3.6)
$$0=\left\{\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}\right\}\{\eta(Y)X-\eta(X)Y\}$$
$$+\eta(Y)QX-\eta(X)QY.$$

Taking inner product on both sides of (3.6) by W, we obtain

(3.7)
$$S(Y,W)\eta(X)=\left\{\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}\right\}\{g(X,W)\eta(Y)-g(Y,W)\eta(X)\}+S(X,W)\eta(Y).$$

Putting $X=\xi$ in (3.7) and using (2.1), (2.3) and (2.14), we get

(3.8)
$$S(Y,W)=\left\{-\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}\right\}g(Y,W)+\left\{-\frac{2n(6n^2+7n+2)-(3n+2)r}{2n(n+1)}\right\}\eta(Y)\eta(W).$$

From (3.8), we have

(3.9)
$$S(Y,W)=Ag(Y,W)+B\eta(Y)\eta(W),$$

where $A=\left\{-\frac{2n(4n^2+5n+2)-(3n+2)r}{2n(n+1)}\right\}$ and $B=\left\{-\frac{2n(6n^2+7n+2)-(3n+2)r}{2n(n+1)}\right\}$. Hence the manifold is an η-Einstein manifold. This completes the proof of the theorem.

Theorem 3.3 A contact conformally semi-symmetric LP-Sasakian manifold (M^{2n+1},g) is an Einstein manifold and a manifold of constant curvature $r=2n(2n+1)$.

Proof: Let us consider an LP-Sasakian manifold (M^{2n+1},g) satisfying the condition $R(X,Y).C_0=0$. Now, we have

(3.10)
$$(R(X,Y).C_0)(U,V)Z=R(X,Y)C_0(U,V)Z-C_0(R(X,Y)U,V)Z-C_0(U,R(X,Y)V)Z-C_0(U,V)R(X,Y)Z.$$

In view of (2.20) and (3.10), we get

(3.11)
$$0=R(X,Y)C_0(U,V)Z-C_0(R(X,Y)U,V)Z-C_0(U,R(X,Y)V)Z-C_0(U,V)R(X,Y)Z.$$

Taking $X=\xi$ in (3.11) and using (2.11), we obtain

(3.12)
$$0=g(C_0(U,V)Z,Y)\xi-\eta(C_0(U,V)Z)Y-g(Y,U)C_0(\xi,V)Z+\eta(U)C_0(Y,V)Z-g(Y,V)C_0(U,\xi)Z+\eta(V)C_0(U,Y)Z-g(Y,Z)C_0(U,V)\xi+\eta(Z)C_0(U,V)Y.$$

Taking inner product on both sides of (3.12) by ξ, we get

(3.13)
$$0=-g(C_0(U,V)Z,Y)-\eta(C_0(U,V)Z)\eta(Y)-g(Y,U)\eta(C_0(\xi,V)Z)+\eta(U)\eta(C_0(Y,V)Z)-g(Y,V)\eta(C_0(U,\xi)Z)+\eta(V)\eta(C_0(U,Y)Z)-g(Y,Z)\eta(C_0(U,V)\xi)+\eta(Z)\eta(C_0(U,V)Y).$$

Putting $Y=U$ in (3.13) we obtain

(3.14)
$$0=g(C_0(U,V)Z,U)+g(U,U)\eta(C_0(\xi,V)Z)+g(U,V)\eta(C_0(U,\xi)Z)-\eta(V)\eta(C_0(U,U)Z)+g(U,Z)\eta(C_0(U,V)\xi)-\eta(Z)\eta(C_0(U,V)U).$$

Now, from (2.16) we have

(3.15)
$$\eta(C_0(U,V)\xi)=0,$$

(3.16)
$$\eta(C_0(\xi,V)Z) = -\frac{1}{n}S(V,Z) - \frac{2n(4n^2+5n+2)-(3n+2)r}{2n^2(n+1)}g(V,Z) - \frac{2n(6n^2+7n+2)-(3n+2)r}{2n^2(n+1)}\eta(V)\eta(Z)$$

and

(3.17)
$$\eta(C_0(U,U)Z) = 0.$$

By virtue of (3.15) and (3.17), (3.14) reduces to

(3.18)
$$0 = g(C_0(U,V)Z,U) + g(U,U)\eta(C_0(\xi,V)Z) + g(U,V)\eta(C_0(U,\xi)Z) - \eta(Z)\eta(C_0(U,V)U).$$

Let $\{e_i : i = 1,2,..,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $U = e_i$ in (3.18) and taking summation over $i, 1 \le i \le 2n+1,$ we get

(3.19)
$$\sum_{i=1}^{2n+1} g(C_0(e_i,V)Z,e_i) + 2n\eta(C_0(\xi,V)Z) - \sum_{i=1}^{2n+1}\eta(Z)\eta(C_0(e_i,V)e_i) = 0,$$

since $\sum_{i=1}^{2n+1} g(e_i,V)\eta(C_0(e_i,\xi)Z) = -\eta(C_0(\xi,V)Z).$

Again, from (2.16) it follows

(3.20)
$$\sum_{i=1}^{2n+1} g(C_0(e_i,V)Z,e_i) = \frac{2(n+1)}{n}S(V,Z) + \left[-\frac{2n(4n^2-n-2)+(2n^2+n+2)r}{2n^2(n+1)}\right]g(V,Z) + \left[-\frac{4n(2n^3-n^2-5n-2)+(-2n^2+2n+4)r}{2n^2(n+1)}\right]\eta(V)\eta(Z)$$

under the condition $tr.\varphi = tr.(\varphi Q) = 0$ and by the use of (2.2), (2.8) and (2.15). From the definition of contact conformal curvature tensor, we also have

(3.21)
$$\sum_{i=1}^{2n+1}\eta(Z)\eta(C_0(e_i,V)e_i) = \frac{(2n+1)\{r-2n(2n+1)\}}{n(n+1)}\eta(V)\eta(Z).$$

In view of (3.16), (3.20) and (3.21), (3.21) takes the form

(3.22)
$$S(V,Z) = \lambda_1 g(V,Z) + \lambda_2\eta(V)\eta(Z),$$

where $\lambda_1 = \frac{2n(8n^3+14n^2+3n-2)-(4n^2+3n-2)r}{4n(n+1)}$ and $\lambda_2 = \frac{(n-1)\{2n(2n+1)-r\}}{n}$.

Taking an orthonormal frame field at any point of the manifold and contracting over V and Z in (3.22) we get

(3.23)
$$r = 2n(2n+1).$$

Using (3.23) in (3.22) we obtain

(3.24)
$$S(V,Z) = 2ng(V,Z).$$

In view of (3.23) and (3.24), the theorem is proved.

4. Conclusions

In this paper, we have studied on contact conformal curvature tensor in a $(2n+1)$-dimensional Lorentzian para-Sasakian manifold (briefly, LP-Sasakian manifold). We have investigated that contact conformally flat and ξ-contact conformally flat LP-Sasakian manifold is an η-Einstein manifold. It is also proved that a contact conformally semi-symmetric LP-Sasakian manifold is an Einstein manifold.

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