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Quaternions, 2x2 complex matrices and Lorentz transformations

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Abstract

We show that the matrix method to construct Lorentz transformations permit to deduce the corresponding quaternionic procedure.

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1. Introduction

In spacetime an event is represented by $\left(x^{j}\right)=(ct,x,y,z)$, $j=0,\ldots,3$, with the metric $\left(g_{jr}\right)=\text{Diag}\,(1,-1,-1,-1)$. If it is necessary to employ another frame of reference, then the new coordinates $\tilde{x}^{r}$ are connected with x^{j} via the linear transformation:

$$\tilde{x}^{j}=L^{j}{}_{r}x^{r} \quad , \tag{1}$$

where the Lorentz's matrix $\underset{\sim}{L}$ verifies the restriction:

$$L^{j}{}_{a}\, g_{rj}L^{r}{}_{b}=g_{ab} \quad , \tag{2}$$

because the Minkowskian line element must remain invariant under $\underset{\sim}{L}$, that is, $\tilde{x}^{r}\tilde{x}_{r}=x^{r}x_{r}$

From (2) we see that $\underset{\sim}{L}$ has six degrees of freedom, which permits to work with four complex numbers $\alpha,\beta,\gamma,\delta$ such that $\alpha\delta-\beta\gamma=1$, then the components of homogeneous Lorentz transformation $\underset{\sim}{L}$ can be written in the form [1-5] (i = √-1):

$$L^{0}{}_{0}=\frac{1}{2}\left(\alpha\alpha^{*}+\beta\beta^{*}+\gamma\gamma^{*}+\delta\delta^{*}\right) \quad , \quad L^{0}{}_{1}=\frac{1}{2}\left(\alpha^{*}\beta+\gamma^{*}\delta\right) \ + \quad \text{c.c.} \ ,$$

$$
\begin{aligned}
L^0{}_2 &= \frac{i}{2}\left(\alpha^*\beta + \gamma^*\delta\right) + \text{c.c.}\ , \quad L^0{}_3 = \frac{1}{2}\left(\alpha\alpha^* - \beta\beta^* + \gamma\gamma^* - \delta\delta^*\right)\ ,\\
L^1{}_0 &= \frac{1}{2}\left(\alpha^*\gamma + \beta^*\delta\right) + \text{c.c.}\ , \quad L^1{}_1 = \frac{1}{2}\left(\alpha^*\delta + \beta\gamma^*\right) + \text{c.c.}\ ,\\
L^1{}_2 &= \frac{i}{2}\left(\alpha^*\delta + \beta\gamma^*\right) + \text{c.c.}\ , \quad L^1{}_3 = \frac{1}{2}\left(\alpha^*\gamma - \beta^*\delta\right) + \text{c.c.}\\
L^2{}_0 &= \frac{i}{2}\left(\alpha\gamma^* - \beta^*\delta\right) + \text{c.c.}\ , \quad L^2{}_1 = \frac{i}{2}\left(\alpha\delta^* + \beta\gamma^*\right) + \text{c.c.}\ ,\\
L^2{}_2 &= \frac{1}{2}\left(\alpha^*\delta - \beta^*\gamma\right) + \text{c.c.}\ , \quad L^2{}_3 = \frac{i}{2}\left(\alpha\gamma^* + \beta^*\delta\right) + \text{c.c.}\ ,\\
L^3{}_0 &= \frac{1}{2}\left(\alpha\alpha^* + \beta\beta^* - \gamma\gamma^* - \delta\delta^*\right)\ , \quad L^3{}_1 = \frac{1}{2}\left(\alpha^*\beta - \gamma^*\delta\right) + \text{c.c.}\ ,\\
L^3{}_2 &= \frac{i}{2}\left(\alpha^*\beta - \gamma^*\delta\right) + \text{c.c.}\ , \quad L^3{}_3 = \frac{1}{2}\left(\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \delta\delta^*\right)\ ,
\end{aligned}
\tag{3}
$$

where c.c. means the complex conjugate of all the previous terms.
Therefore, any complex 2x2 matrix:

$$
\underset{\sim}{U} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\ , \quad \text{Det}\ \underset{\sim}{U} = \alpha\delta - \beta\gamma = 1\ , \tag{4}
$$

generates one Lorentz matrix through (3), or via the expression [6-8]:

$$
\begin{pmatrix} \tilde{x}^o + \tilde{x}^3 & \tilde{x}^1 - i\,\tilde{x}^2 \\ \tilde{x}^1 + i\tilde{x}^2 & \tilde{x}^o - \tilde{x}^3 \end{pmatrix} = \underset{\sim}{U} \begin{pmatrix} x^o + x^3 & x^1 - i\,x^2 \\ x^1 + ix^2 & x^o - x^3 \end{pmatrix} \underset{\sim}{U}^\dagger\ , \tag{5}
$$

where:

$$
\underset{\sim}{U}^\dagger = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\ , \quad \text{Det}\ \underset{\sim}{U}^\dagger = \left(\ \text{Det}\ \underset{\sim}{U}\ \right)^* = 1\ , \tag{6}
$$

thus the determinant of (5) implies the invariance of the line element:

$$
\left(\tilde{x}^o\right)^2 - \left(\tilde{x}^1\right)^2 - \left(\tilde{x}^2\right)^2 - \left(\tilde{x}^3\right)^2 = \left(x^o\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2\ . \tag{7}
$$

In the next Section, from (5) we shall deduce a quaternionic expression to produce Lorentz transformations.

2. Lorentz's matrix via quaternions

The matrix (4) can be written in the form [9, 10]:

$$\underset{\sim}{U} = a_0 \underset{\sim}{I} - ia_1\sigma_1 - ia_2\sigma_2 - ia_3\sigma_3 \quad , \tag{8}$$

being σ_j the Cayley [11]-Sylvester [12]-Pauli matrices [13]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{9}$$

with the Euler-Olinde Rodrigues parameters [14]:

$$\begin{aligned} a_0 &= \frac{1}{2}(\alpha + \delta) \quad , \quad a_1 = \frac{i}{2}(\gamma + \beta) \quad , \\ a_2 &= \frac{1}{2}(\gamma - \beta) \quad , \quad a_3 = \frac{i}{2}(\alpha - \delta) \quad , \end{aligned} \tag{10}$$

If we employ the following formal association with the quaternionic units [15]:

$$\underset{\sim}{I} \to 1 \quad , \quad -i\sigma_1 \to \mathbf{I} \quad , \quad -i\sigma_2 \to \mathbf{J} \quad , \quad -i\sigma_3 \to \mathbf{K} \quad , \tag{11}$$

then the properties of the σ_j give us the rules of multiplication :

$$\mathbf{I^2} = \mathbf{J^2} = \mathbf{K^2} = -1 \quad , \quad \mathbf{I\,J} = \mathbf{K} \quad , \quad \mathbf{J\,K} = \mathbf{I} \quad , \quad \mathbf{K\,I} = \mathbf{J} \quad , \tag{12}$$

besides from (8) it appears the relationship:

$$\underset{\sim}{U} \to \mathbf{A} = a_0 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K} \quad , \tag{13}$$

where **A** is a unitary quaternion because the condition $\alpha\delta - \beta\gamma = 1$ and (10) imply:

$$\mathbf{A\overline{A}} = a_0^{\ 2} + a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2} = 1 \quad , \tag{14}$$

with

$$\overline{\mathbf{A}} = a_0 - a_1\mathbf{I} - a_2\mathbf{J} - a_3\mathbf{K} \ . \tag{15}$$

Therefore, any 2x2 complex matrix (4) has associated a unitary quaternion via (10) and (13). Now we shall apply this procedure to each matrix appearing in (5) to obtain its quaternionic version:

$$\begin{aligned} \begin{pmatrix} x^o + x^3 & x^1 - i\,x^2 \\ x^1 + ix^2 & x^o - x^3 \end{pmatrix} &\to \mathbf{R} = x^0 + ix^1\mathbf{I} + ix^2\mathbf{J} + ix^3\mathbf{K} \quad , \\ &= \mathrm{ct} + ix\mathbf{I} + iy\mathbf{J} + iz\mathbf{K} \quad , \\ \underset{\sim}{U} \to \mathbf{A} \quad , \ \underset{\sim}{U}^{\dagger} &\to \overline{\mathbf{A}}^* = a_0^* - a_1^*\mathbf{I} - a_2^*\mathbf{J} - a_3^*\mathbf{K} \quad , \end{aligned} \tag{16}$$

thus (5) implies [16-21]:

$$\tilde{\mathbf{R}} = \mathbf{A}\mathbf{R}\overline{\mathbf{A}}^{*}, \tag{17}$$

which generates the linear mapping (1) and reproduces all the components (3) because (10) gives the link:

$$\begin{aligned} \alpha &= a_0 - ia_3 \quad , \quad \beta = -a_2 - ia_1 \quad , \\ \gamma &= a_2 - ia_1 \quad , \quad \delta = a_0 + ia_3 \quad ; \end{aligned} \tag{18}$$

Then (17) is a quaternionic factory to elaborate Lorentz transformations.

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