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Some special characterisations of Fredholm operators in Banach space

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Abstract

A bounded linear operator which has a finite index and which is defined on a Banach space is often referred to in the literature as a Fredholm operator. Fredholm operators are important for a variety of reasons, one being the role that their index plays in global analysis. The aim of this paper is to prove the spectral theorem for compact operators in refined form and to describe some properties of the essential spectrum of general bounded operators by the use of the theorem of Fredholm operators. For this, we have analysed the Fredholm operator which is defined in a Banach space for some special characterisations.

Keywords: Bounded linear operator; Compact operator; Fredholm operator; Banach space.

1. Introduction

An operator K defined by a kernel k is called a Fredholm type operator. The name goes back to Swede, E. Ivar Fredholm who developed a comprehensive theory for integral equations of second kind at the beginning of the twentieth century [4].

Let X = Y = C[a,b] be a Banach Space.Let k(s,t) be defined for $a \leq s \leq b$and $a \leq t \leq b$. Then for each $x \epsilon X$ the Riemann integral

$$\int_a^b k(s,t)x(t)dt \qquad (1)$$

exists and defines a continuous function of s on [a,b].

The integral (1) defines a linear operator K on X into X. If we take $Kx = y$ to mean

$$y(s) = \int_a^b k(s,t)x(t)dt \qquad (2)$$

the equation(2) is known as Fredholm type integral equation of the first kind [7].

Another operator T is obtained by defining

$$Tx = y$$

to mean

$$y(s) = x(s) - \int_a^b k(s,t)x(t)dt \qquad (3)$$

Here k(s,t) is a continuous function on [a,b]×[a,b] and is called the kernel of the integral equation. y(s) is continuous on [a,b] and therefore y∈C[a,b]. The equation(3) is known as Fredholm type integral equation of the second kind [5].

Equations of this sort are of great importance. The application of this operator plays a vital role in the theory of boundary value problems in differential equations. Fredholm studied Fredholm type integral equations of the second kind, which gave rise to such operators. In view of the development of the theory of Fredholm operators the following definitions are frequently used.

Definitions

Fredholm operator

(i) A closed linear operator which has a finite index is called a Fredholm operator.

(ii) Let *X* and *Y* are Banach spaces. A linear operator *T* from *X* to *Y* is called a Fredholmoperator if

i. *T* is closed.
ii. The domain of *T* is dense in *X*.
iii. *α(T)*, the dimension of the null space *N(T)* of *T* is finite.
iv. The range of *T* is closed in *y*
v. *B(T)*, The co-dimension of *R(T)* in *Y* is finite.

The terminology stems from the classical theory of integral equations. Special types of Fredholm operators were considered by many authors since that time but systematic treatment were not given until the work of Atkimon [1], Gohberg [4] and yood [9]. A general account of the history of the theory is given by [a]Gohberg Krein [3] and (b) Kato [5]. For a good general account of the theory can be found in the book written by Gohberg [4].

Definition(3)

Let B & C be two Banach spaces. A bounded linear operator A: B → C is defined to be a linear map for which the norm

$$\| A \| := \sup\{\| Af \| : \| f \| \leq 1\}$$

is finite.

Definition(4)

Let X and y be normed linear spaces. Suppose T is a linear operator with domain X and range in y. We say that T is compact if for each bounded sequence {xn} in X, the sequence {T xn} contains a sub sequence converging to some limit in y. A compact operator is also called completely continuous.

Lemma (1)

If A is a compact operator on B, then (λI- A) is Fredholm for all λ ≠ 0.

Proof:

We first prove that $\mathcal{L}$□Ker(λI - A) is finite dimensional by contradiction. If this were not the case there would exist an infinite sequence $x_n \in \mathcal{L}$ such that $\| x_n \| = 1$ and $\| x_m - x_n \| \geq 1/2$ for all distinct m and n. Since Axn = λxn and λ ≠ 0, we could conclude that Axn has no convergent subsequence. We can write B=$\mathcal{L}$ + M, where $\mathcal{L} \cap M = \{0\}$and M is a closed linear subspace on which (λI - A) is one-one.

We next prove that $\mathcal{R} := \mathrm{Ran}(\lambda I - A)$ is closed. If $g_n \in \mathcal{R}$and $\| g_n - g \| \to 0$, then there exist $f_n \in M$ such that $g_n = (\lambda I - A)f_n$. If $\| f_n \|$ is not a bounded sequence then by passing to a subsequence (without change of notation) we may assume that $\| f_n \| \to \infty$ as $n \to \infty$. Putting $h_n := f_n / \| f_n \|$ we have $\| h_n \| = 1$ and $k_n := (\lambda I - A)h_n \to 0$. The compactness of A implies that $h_n = \lambda^{-1}(Ah_n + k_n)$ has a convergent subsequence. Passing to this subsequence we have $h_n \to h$ where $\| h \| = 1$, $h \in M$, and $h = \lambda^{-1}Ah$. We conclude that $h \in M \cap \mathcal{L}$. The contradiction implies that $\| f_n \|$ is a bounded sequence. Given this fact the compactness of A implies that the sequence $f_n = \lambda^{-1}(Af_n + g_n)$ has a convergent subsequence. Passing to this subsequence we obtain $f_n \to f$ as $n \to \infty$, so $f = \lambda^{-1}(Af + g)$, and $g = (\lambda I - A)f$. Therefore $\mathcal{R}$ is closed.

Since Ran$(\lambda I - A)$ is closed, an application of the Hahn-Banach theorem implies that its codimension equals the dimension of Ker$(\lambda I - A^*)$ in B*. But A^* is compact, so this is finite by the first paragraph.

Our next theorem provides a second characterization of Fredholm operators.

Theorem (1)

Every Fredholm operator has closed range. The bounded operator $A : B \to C$ is Fredholm if and only if there is a bounded operator $B : C \to B$ such that both $(AB - I)$ and $(BA - I)$ are compact.

Proof:

If A is Fredholm then $B_1 :=$ Ker(A) is finite-dimensional and so has a complementary closed subspace B_0 in B. Moreover A maps B_0 one-one onto Co $:=$ Ran(A). If C_1 is a complementary finite-dimensional subspace of C_0 in C then the operator $X : B_0 \oplus C_1 \to C$ defined by

$$X(f \oplus v) := Af + v$$

is bounded and invertible. We deduce by the inverse mapping theorem that $C_0 := X(B_0)$ is closed. This completes the proof of the first statement of the theorem.

Still assuming that A is Fredholm, put $B(g \oplus v) := (A_0)^{-1}g$ for all $g \in C_0$ and $v \in C_1$, where $A_0 : B_0 \to C_0$ is the restriction of A to B_0. Then B is a bounded operator from C to B and both of

$$K_1 := AB - I, \quad K_2 := BA - I \qquad (1)$$

are finite rank and hence compact.

Conversely suppose that A, B are bounded, K_1, K_2 are compact and (1) hold. Then

$$Ker(A) \subseteq Ker(I + K_2),$$
$$Ran(A) \supseteq Ran(I + K_1).$$

Since $(I + K_1)$ and $(I + K_2)$ are both Fredholm by Lemma (7.1), it follows that A must be Fredholm.

The proof of Theorem (1) provides an important structure theorem for Fredholm operators.

Theorem (2)

If A is a Fredholm operator then there exist decompositions $B = B_0 \oplus B_1$ and $C = C_0 \oplus C_1$ such that

(i) B_0 and C_0 are closed subspaces;

(ii) B_1 and C_1 are finite-dimensional subspaces;

(iii) B_1 = Ker(A) and C_0= Ran(A);

(iv) Index(A) = dim(B_1) - dim(C_1);

(v) A has the matrix representation

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad (2)$$

where $A_0 : B_0 \rightarrow C_0$ is one-one onto.

Example (1)

Let A be a Fredholm operator on the Banach space B. Prove that if Ker(A) = {0} then Ker(A - □I) = {0} for all small enough □. It can be proved that if Ran(A) = B then Ran(A - □I) = B for all small enough □.

Before stating our next theorem we make some definitions. We say that λ lies in the essential spectrum EssSpec(A) of a bounded operator A if (λI - A) is not a Fredholm operator.

Since the set κ (B) of all compact operators is a norm closed two-sided ideal in the Banach algebra $\mathcal{L}$(B) of all bounded operators on B, the quotient algebra C := $\mathcal{L}$(B)/ κ(B) is a Banach algebra with respect to the quotient norm

$$\| \pi(A) \| := \inf\{\| A + K \| : K \in \kappa(B)\}$$

where $\pi : \mathcal{L}(\mathrm{B}) \rightarrow$ C is the quotient map. The Calkin algebra C enables us to rewrite Theorem (1) is particularly in a simple form.

Theorem (3)

The bounded operator A on B is Fredholm if and only if $\pi(A)$ is invertible in the Calkin algebra C. If A Є $\mathcal{L}$(B) then

$$EssSpec(A) = Spec(\pi(A)).$$

Proof :

Both statements of the theorem are elementary consequences of Theorem (1).

Corollary:

If A : B → B is a Fredholm operator and $B := A + K$ where K is compact, then B is a Fredholm operator and

$$EssSpec(A) = EssSpec(B).$$

Theorem (4)

If A is a Fredholm operator on B then A^* is Fredholm.

Proof :

Suppose that $AB = I + K_1$ and $BA = I + K_2$ where K_1, K_2 are compact. Then $B^*A^* = I + K_1^*$ and $A^*B^* = I + K_2^*$. We deduce that A^* is Fredholm by applying Theorem (1).

Example (2):

Prove directly from the definition that if A_1 and A_2 are both Fredholm operators then so is A_1A_2.
Note: If $B_1 = B_2$ then this is an obvious consequence of Theorem (7.3), but there is an elementary direct proof.

Theorem (5)

If A:B → C is a Fredholm operator, then there exists Є >0 such that every bounded operator X satisfying $\| X - A \| < \in$ is also Fredholm with

$$index(X) = index(A).$$

Proof:
We make use of the matrix representation of Theorem (2). If

$$X = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

and $\| X - A \| <$ Є then $\| B - A_0 \| < c\epsilon$, so B is invertible provided $\epsilon > 0$ is small enough.

If $f \epsilon B_O$ and $g \epsilon B_1$ then $X(f \oplus g) = 0$ if and only if

$$Bf + Cg = 0,$$
$$Df + Eg = 0.$$

This reduces to

$$(E - DB^{-1}C)g = 0,$$

where $(E - DB^{-1} C)g : B_1 \to C_1$, both of these spaces being finite-dimensional. We deduce that

$$dim(Ker(X)) = dim(Ker(E - DB^{-1}C))$$

for all small enough Є > 0. By applying a similar argument to

$$X^* = \begin{pmatrix} B^* & D^* \\ C^* & E^* \end{pmatrix}$$

we obtain

$$dim(Coker(X)) = dim(Coker(E - DB^{-1}C))$$

for all small enough ε > 0.

Problem now implies that

$$index(X) = index(E - DB^{-1}C)) = dim(\mathcal{B}_1) - dim(C_1).$$

This formula establishes that index(X) does not depend on X, provided $\| X - A \|$ is small enough.

Theorem (5) establishes that the index is a homotopy invariant: if $t \to A_t$ is a norm continuous family of Fredholm operators then index(A_t) does not depend on t. In a Hilbert space context one can even identify the homotopy classes.

Conclusion

Thus, we see that Fredholm operators are important for variety of reasons, one being the role that their index plays in global analysis. The dimension of null space N(T) and the co-dimension of R(T) of the operator T are finite and it is closed and range of the operator is also closed. The application of this operator plays a vital role in the theory of boundary value problems in differential equations.

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