Stability of the Sitnikov's circular restricted three body problem when the primaries are oblate spheroid

R.R. Thapa

Department of Mathematics, Post Graduate Campus, Biratnagar
Tribhuvan University, Nepal
Email: thaparajuram@yahoo.com
Accepted for publication: February 06, 2014

Abstract

The Sitnikov's problem is a special case of restricted three body problem if the primaries are of equal masses ($m_1 = m_2 = 1/2$) moving in circular orbits under Newtonian force of attraction and the third body of mass m_3 moves along the line perpendicular to plane of motion of primaries. Here oblate spheroid primaries are taken. The solution of the Sitnikov's circular restricted three body problem has been checked when the primaries are oblate spheroid. We observed that solution is depended on oblate parameter A of the primaries and independent variable $\tau = \eta t$. For this the stability of non-trivial solutions with the characteristic equation is studied. The general equation of motion of the infinitesimal mass under mutual gravitational field of two oblate primaries are seen at equilibrium points. Then the stability of infinitesimal third body m_3 has been calculated.

Keywords: Characteristic equation; Non-trivial solution; Oblate spheroid; Sitnikov's restricted three body problem; Stability.

1. Introduction

The primaries in the Sitnikov's problem move on the circumference of the same circle if the primaries are spheroid in shape. But if the primaries are oblate spheroid in shape then due to their oblateness the primaries will not be equidistant from their center of mass. The primaries should be equidistant from their centre of mass. For this Principal axes of the oblate bodies should be parallel to the synodic axes andThe masses of both the primaries should be equal.

The system consists of two oblate primaries with equal masses ($m_1 = m_2 = 1/2$). The third body of mass m_3 is much less than the masses of the primaries. Thus the potential between two bodies m_1 and m_2 is given by

$$\Rightarrow -v = \frac{Gm_1 m_2}{r} + \frac{GA(m_1 + m_2)}{r^3}$$

$$A = \frac{a^2 - c^2}{10} = \text{oblatenless of the primaries}$$

where a_i, b_i, c_i ($i = 1, 2$) be the semi axes of the primaries. When the primaries are oblate, $a_i = b_i$.

An integrable case in restricted problem of three bodies by imposing further restrictions on the restricted three body problem, by supposing the two finite bodies of equal masses and an infinitesimal body moving in their common axis of revolution has been studied by MacMillan [1].

In three body problem two primaries of equal masses and these two moving in (i) circular orbits and (ii) elliptic orbits around their center of mass has obtained by Sitnikov [2]. The third body (infinitesimal mass) is moving along a line which is perpendicular to the plane of primaries and passing through the center of mass of the primaries.

The stability and bifurcation of Sitnikov motion have studied by Perdios et.al. [3]. The family of periodic Sitnikov motions can be recognized as a special case ($\mu =1/2$) of the family L_1^3 of three dimensional orbits emanating at L_1 for all admissible values of μ and is denoted by L_1^1. The existence of such a family offers the opportunity to study the stability of bifurcations of one-dimensional motions under perturbations in two additional dimensions which tends to take the moving particle directly into the full three-dimensional space.

The stability of motion in the Sitnikov problem have studied by Soulis et al. [4]. They have varied the mass of the infinitesimal body $0 < m < 10^{-3}$. They have found that as m_3 increases, the domain of allowed motion grows significantly and chaotic regions in the phase space appear through a series of saddle-node bifurcations.

The equations of motions have the non-trivial solutions obtained by Mccuskey [5]

$$\Rightarrow n^2 = \frac{1}{r^3} + \frac{8A}{r^5}$$

The equation of motion of the third body has obtained by Suraj et al. [6], as follows

$$\Rightarrow \frac{d^2 z}{dt^2} + \frac{z}{(z^2 + r^2)^{3/2}} + \frac{9Az}{(z^2 + r^2)^{5/2}} - \frac{15Az^2}{(z^2 + r^2)^{7/2}} = 0$$

2. Solutions By Lindsted-Poincare Method

Equation of motion of third body is

$$\frac{d^2 z}{dt^2} + \frac{z}{\left(z^2 + r^2\right)^{3/2}} + \frac{9Az}{\left(z^2 + r^2\right)^{5/2}} - \frac{15Az^2}{\left(z^2 + r^2\right)^{7/2}} = 0$$

Raju Ram Thapa, et al. [7] has obtained the solution as follows:

$$z = c\cos\eta t + \frac{3}{64}\left(\cos\eta t - \cos 3\eta t\right)c^3 + \frac{9c^5}{4096}\left(23\cos\eta t - 24\cos 3\eta t + \cos 5\eta t\right)$$
$$+\frac{27c^7}{8}\left[\frac{547}{32768}\cos\eta t - \frac{297}{16384}\cos 3\eta t + \frac{3}{2048}\cos 5\eta t - \frac{1}{32768}\cos 7\eta t\right]$$
$$+A\left[\frac{1}{32}\left(\cos\eta t - \cos 3\eta t\right)c^3 + \frac{3}{1024}\left(23\cos\eta t - 24\cos 3\eta t + \cos 5\eta t\right)c^5\right.$$
$$\left.+\frac{27}{4}\left(\frac{547}{32768}\cos\eta t - \frac{297}{16384}\cos 3\eta t + \frac{3}{2048}\cos 5\eta t - \frac{1}{32768}\cos 7\eta t\right)c^7 + ...\right]$$

3. Stability:

The general equations of motion of the infinitesimal mass under the mutual gravitational field of two oblate primaries are

$$
\left.\begin{aligned}
\ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x} \\[2mm]
\ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y} \\[2mm]
\ddot{z} &= \frac{\partial \Omega}{\partial z}
\end{aligned}\right\} \tag{1}
$$

where *n* is the mean angular velocity of the primaries moving on the common circular orbit of radius *a*/2.

Here,

$$
\Omega = \frac{1}{(z^2 + r^2)^{\frac{1}{2}}} + \frac{A}{(z^2 + r^2)^{\frac{3}{2}}} - \frac{3Az^2}{(z^2 + r^2)^{\frac{5}{2}}} \tag{2}
$$

$$
b = 1 + \frac{16}{3}A
$$

A = oblateness parameter of the primaries.

Hence from (2), it is clear that Ω is independent of *x* and *y* and depends only upon *z* alone hence

$\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial y} = 0.$

$$
\frac{\partial \Omega}{\partial z} = -\frac{z}{(z^2 + b)^{\frac{3}{2}}} - \frac{9Az}{(z^2 + b)^{\frac{5}{2}}} + \frac{15Az^3}{(z^2 + b)^{\frac{7}{2}}} \tag{3}
$$

For the equilibrium solution of the restricted three body problem,

$\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial y} = \frac{\partial \Omega}{\partial z} = 0.$

Thus the system of equations (1) can be written as

$$
\left.\begin{aligned}
\ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x} = f(x, y, z) \text{ (say)} \\[2mm]
\ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y} = g(x, y, z) \text{ (say)} \\[2mm]
\ddot{z} &= \frac{\partial \Omega}{\partial z} = h(x, y, z) \text{ (say)}
\end{aligned}\right\} \text{ at the equilibrium points} \tag{4}
$$

Let us denote the equilibrium point as $L(0, 0, z_0)$, we shall now communicate small displacement ξ, η, ζ in the coordinates of *L* such that $x = 0 + \xi$, $y = 0 + \eta$, $z = z_0 + \zeta$ satisfy the above equation.

Thus,

$$\Rightarrow \frac{d^2\xi}{dt^2} - 2n\dot{\eta} = f(0+\xi, 0+\eta, z_0+\zeta)$$

$$\frac{d^2\eta}{dt^2} - 2n\dot{\xi} = g(0+\xi, 0+\eta, z_0+\zeta)$$

$$\frac{d^2\zeta}{dt^2} = h(0+\xi, 0+\eta, z_0+\zeta) \text{ where } \frac{d^2 z_0}{dt^2} = 0.$$

$$\Rightarrow \ddot{\xi} - 2n\dot{\eta} = f(0,0,z_0) + \xi\left(\frac{\partial f}{\partial x}\right)_0 + \eta\left(\frac{\partial f}{\partial y}\right)_0 + \zeta\left(\frac{\partial f}{\partial z}\right)_0 + \text{higher order infinitesimals}$$

$$\ddot{\eta} + 2n\dot{\xi} = g(0,0,z_0) + \xi\left(\frac{\partial g}{\partial x}\right)_0 + \eta\left(\frac{\partial g}{\partial y}\right)_0 + \zeta\left(\frac{\partial g}{\partial z}\right)_0 + \text{higher order infinitesimals}$$

$$\ddot{\zeta} = h(0,0,z_0) + \xi\left(\frac{\partial h}{\partial x}\right)_0 + \eta\left(\frac{\partial h}{\partial y}\right)_0 + \zeta\left(\frac{\partial h}{\partial z}\right)_0 + \text{higher order infinitesimals}$$

Since $(0, 0, z_0)$ is a solution of the set of equation (4)

$$\frac{\partial\Omega}{\partial x} = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial z} = 0, \text{ hence}$$

$f(0, 0, z_0) = g(0, 0, z_0) = h(0, 0, z_0) = 0$

$$\ddot{\xi} - 2n\dot{\eta} = \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial x}\right)_0 + \eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial x}\right)_0 + \zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial x}\right)_0 + \text{higher order infinitesimals}$$

Thus,

$$\ddot{\eta} + 2n\dot{\xi} = \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial y}\right)_0 + \eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial y}\right)_0 + \zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial y}\right)_0 + \text{higher order infinitesimals}$$

$$\ddot{\zeta} = \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial z}\right)_0 + \eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial z}\right)_0 + \zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial z}\right)_0 + \text{higher order infinitesimals}$$

Neglecting higher order terms of ξ, η, ζ, we get the variational equations as

$$\left.\begin{aligned}
\Rightarrow \ddot{\xi} - 2n\dot{\eta} &= \xi\Omega^0_{xx} + \eta\Omega^0_{yx} + \zeta\Omega^0_{zx} \\
\ddot{\eta} + 2n\dot{\xi} &= \xi\Omega^0_{xy} + \eta\Omega^0_{yy} + \zeta\Omega^0_{zy} \\
\ddot{\zeta} &= \xi\Omega^0_{xz} + \eta\Omega^0_{yz} + \zeta\Omega^0_{zz}
\end{aligned}\right\} \tag{5}$$

where $\Omega^0_{xx}, \Omega^0_{xy}, \Omega^0_{xz}, \ldots$ etc represent the value of $\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial x}\right)_0, \frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial y}\right)_0, \frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial z}\right)_0, \ldots$ at the equilibrium

points.

Now the system of equations (5) can be written in the form of a single matrix equation as

$$
\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \\ \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \end{bmatrix}_{6\times 1}
=
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\Omega^0_{xx} & \Omega^0_{yx} & \Omega^0_{zx} & 0 & 2n & 0 \\
\Omega^0_{xy} & \Omega^0_{yy} & \Omega^0_{zy} & -2n & 0 & 0 \\
\Omega^0_{zx} & \Omega^0_{zy} & \Omega^0_{zz} & 0 & 0 & 0
\end{bmatrix}_{6\times 6}
\begin{bmatrix} \xi \\ \eta \\ \zeta \\ \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix}_{6\times 1}
\qquad (6)
$$

$$\text{i.e. } \dot{X} = AX \qquad\qquad\qquad (7)$$

when $\begin{bmatrix} \xi & \eta & \zeta & \dot{\xi} & \dot{\eta} & \dot{\zeta} \end{bmatrix}^T$ is a 6×1 column matrix of derivatives of the variations of the coordinates of the equilibrium points.

$$
A =
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\Omega^0_{xx} & \Omega^0_{yx} & \Omega^0_{zx} & 0 & 2n & 0 \\
\Omega^0_{xy} & \Omega^0_{yy} & \Omega^0_{zy} & -2n & 0 & 0 \\
\Omega^0_{zx} & \Omega^0_{zy} & \Omega^0_{zz} & 0 & 0 & 0
\end{bmatrix}_{6\times 6}
$$

is a 6×6 square matrix of constants.

$$\Omega^0_{xx} = \Omega^0_{yx} = \Omega^0_{zx} = 0$$

But $\quad \dfrac{\partial \Omega}{\partial x} = \dfrac{\partial \Omega}{\partial y} = 0,\ $ so $\quad \Omega^0_{xy} = \Omega^0_{yy} = \Omega^0_{zy} = 0$

$$\Omega^0_{zx} = \Omega^0_{zy} = 0 \text{ and } \Omega^0_{zz} \neq 0$$

$$
A =
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 2n & 0 \\
0 & 0 & 0 & -2n & 0 & 0 \\
0 & 0 & \Omega^0_{zz} & 0 & 0 & 0
\end{bmatrix}
$$

If any matrix X satisfies the equation $AX = \lambda X$ $\hspace{2cm}$ (8)

The first in the solution of (7) is to found as the value of X.

We can write (8) as

$\hspace{2cm}(A - \lambda I)\, X = 0$ $\hspace{4cm}$ (9)

where I is the 6×6 unit matrix.

The set of six simultaneous linear equations in six unknowns $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta}$ will have non-trivial solutions provided the determinant of the matrix $(A - \lambda I)$ vanishes. i.e.

$\hspace{2cm}\left| A - \lambda I \right| = 0$ $\hspace{4cm}$ (10)

To check the stability of the (10) non-trivial solutions. Let us find values of λ from (10) i.e. from

$\hspace{2cm}\Rightarrow \lambda^2 (\lambda^2 + 4n^2)(\lambda^2 - \Omega_{zz}^0) = 0$ $\hspace{3cm}$ (11)

i.e. there are six values of λ. Let $\lambda_1^2, \lambda_2^2, \lambda_3^2$ be three roots corresponding each factor of (11). The conditions for stable solution are

$\hspace{2cm}\lambda_i^2 \leq 0$ $\hspace{2cm}$ for $i = 1, 2, 3$.

From (11)

$$\left.\begin{array}{l} \lambda_1^2 = 0 \Rightarrow \lambda_{11} = \lambda_{12} = 0 \\[2mm] \Rightarrow \lambda_2 = \pm 2ni \text{ (imaginary)} \\[2mm] \Rightarrow \lambda_{21} = 2ni, \quad \lambda_{22} = -2ni \\[2mm] \text{when } \lambda_3^2 - \Omega_{zz}^0 = 0 \Rightarrow \lambda_3 = \pm\sqrt{\Omega_{zz}^0} \end{array}\right\} \hspace{1cm} (12)$$

But $\hspace{1cm}\dfrac{\partial \Omega}{\partial z} = -\dfrac{z}{(z^2 + b^2)^{\frac{3}{2}}} - \dfrac{9Az}{(z^2 + b^2)^{\frac{5}{2}}} + \dfrac{15Az^3}{(z^2 + b^2)^{\frac{7}{2}}}$

$\hspace{1cm}\Rightarrow \dfrac{\partial^2 \Omega}{\partial z^2} = -\dfrac{z}{(z^2 + b^2)^{\frac{3}{2}}} + \dfrac{3z^2}{(z^2 + b^2)^{\frac{5}{2}}} - \dfrac{9A}{(z^2 + b^2)^{\frac{5}{2}}} + \dfrac{90Az^2}{(z^2 + b^2)^{\frac{7}{2}}} - \dfrac{105Az^4}{(z^2 + b^2)^{\frac{9}{2}}}$

$$\Omega^0_{zz} = -\frac{1}{b^3}\left(1-\frac{3}{2}\left(\frac{z_0}{b}\right)^2+...\right)+\frac{3}{b^3}\left(\frac{z_0}{b}\right)^2\left(1-\frac{5}{2}\left(\frac{z_0}{b}\right)^2+...\right)-\frac{9A}{b^5}\left(1-\frac{5}{2}\left(\frac{z_0}{b}\right)^2+...\right)$$

$$+\frac{90A}{b^5}\left(\frac{z_0}{b}\right)^2\left(1-\frac{7}{2}\left(\frac{z_0}{b}\right)^2+...\right)-\frac{105A}{b^5}\left(\frac{z_0}{b}\right)^4\left(1-\frac{9}{2}\left(\frac{z_0}{b}\right)^2+...\right)$$

Since z_0 is very small in comparison of b hence $\frac{z_0}{b}<<1$ and so $\left(\frac{z_0}{b}\right)^2,\left(\frac{z_0}{b}\right)^3,\left(\frac{z_0}{b}\right)^4,...$ are negligible.

Thus we get

$$\Omega^0_{zz} = -\frac{1}{b^3}-\frac{9A}{b^5} = -\left[\frac{1}{\left(1+\frac{16}{3}A\right)^3}+\frac{9A}{\left(1+\frac{16}{3}A\right)^5}\right]$$

$$= -\left[\left(1+\frac{16}{3}A\right)^{-3}+9A\left(1+\frac{16}{3}A\right)^{-5}\right] \text{ using } r^2 = 1+\frac{6A}{3}=b$$

$$= -[1+A]$$

$$\Omega^0_{zz} = -(1+A)$$

$$\Rightarrow \lambda_3^2 = -(1+A)$$

$$\Rightarrow \lambda_3 = \pm\, i\sqrt{1+A}$$

$$\Rightarrow \lambda_{31} = i\sqrt{1+A}; \quad \lambda_{32} = -i\sqrt{1+A}$$

Thus, all the six roots of the characteristic equation are 0, 0; 2ni, -2ni, $i\sqrt{1+A}$; $-i\sqrt{1+A}$ and hence the equilibrium position of the Sitnikov restricted three body problem are stable.

4. Conclusion

The condition of stability is satisfied by solution of the Sitnikov's problem the third body is oscillatory in nature. It displaced slightly from its equilibrium position but stayed in the neighbourhood of the equilibrium point.

Acknowledgments

References

[1] W.D. Macmillan, Astronomical Journal, 27 (1913) 11,

[2] K.A. Stinikov, Dokl. Akad. Nauk, USSR, 133 (1960) 303.

[3] E. Perdois, et al., Celest. Mech. & Dyn. Astron., 42 (1988) 187.

[4] P.S. Soulis, Celest. Mech. & Dyn. Astron., 99 (2007) 129.

[5] S.W. Mccuskey Introduction to Cel. Mechanics, Addison-Wesley publishing company, Inc., 1963.

[6] M.S. Suraj, M.R.Hassan, Proceeding of PAS, 50 (2013) 61.

[7] R.R. Thapa, M.R. Hassan, Proceedings: Modern Trends in Science and Technology, Biratnagar (2013) 129.