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GENERALIZATION OF RIEMANN INTEGRAL: THE LEBESGUE INTEGRAL

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Abstract

This paper deals principally with the two integrals, namely Riemann integral and Lebesgue integral starting with definitions and their existences. Basically the domain of the Riemann integrable function is different from that of the Lebesgue integrable function. It is found that every upper (lower) Riemann integral is greater than (less than) or equal to every upper (lower) Lebesgue integral. It includes that Lebesgue integral is the generalization of Riemann integral and is the focus of this paper.

Key words

Partition; sums; bounds; Reimann integral; Lebesgue integral

Introduction

Calculus deals principally with two geometric problems: finding the tangent line to a curve, and finding the area of a region under a curve. The first is studied by a limit process known as differentiation; the second by another limit process-integration to which we turn now.

The two concepts, derivative and integral, arise in entirely different ways and it is remarkable fact indeed that the two are intimately connected. If we consider the definite integral of a continuous function f as a function of its upper limit, say we write

$$F(x) = \int_a^x f(t)\,dt,$$

then F has a derivative and F'(x) = f(x). This important result shows that differentiation and integration are , in a sense, inverse operations(Apostal,*1994*)

Gottfried Wilhelm Leibniz (1646-1716) could argue that the ordinates to the points on a curve represent infinitesimal rectangles of height y and width dx and hence finding the area under the curve – "summing all the lines in the figure"-amounted to summing infinitesimal differences in area dA, which collapsed to give the total area. Since it was obvious that on the infinitesimal level dA = y dx, the fundamental theorem of calculus was an immediate consequence. Leibniz eventually abbreviated the sum of all the increments in the area (that is, the total area) using an elongated S, so that A = $\int dA$ = $\int y\, dx$ (Cooke, *1997*).

In nineteenth century, Augustin-Louin Cauchy (1789-1857) established calculus on the basis of the modern concept of the limit. He defined the integral as the limit of a sum, rather than in terms of anti-derivatives. Returning to the notion that the area under a function could be approximated by summing together the areas of approximately selected rectangles, Cauchy noted that for continuous functions, the resulting sum became more accurate as more rectangles of smaller width were used. But rather than using this sum as an estimation for the area under a function, he defined the integral as the limit of the sum of rectangles constructed by subdividing an interval, and using the value of the function at endpoints to determine a rectangle's height. Symbolically, it can be represented as

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i\text{-}x_{i-1})$, where x_i represents the right endpoint of each subinterval, for each $1\leq i \leq n$.

Nevertheless, Cauchy's definition guaranteed the existence of the definite integral only for functions with at most a finite number of discontinuities. For this reason, Georg Bernhard Riemann sought to generalize Cauchy's integral so that a wider class of functions could be integrated. Riemann did so by allowing the height of the approximating rectangles to be determined by any point in the corresponding subinterval, rather than merely by the endpoints. Thus, Riemann's integral took the form of

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)(x_i\text{-}x_{i-1})$, where t_i represents a sample point taken from the interval $[x_{i-1},x_i]$ (Wells, *2011*).

Henry lebesgue (1875-1941) established the theory of integration, which was a generation of the 17th century concept of integration – summing the area between an axis and the curve of a function defined for that axis

Lebesgue integral can be seen as a generalization of the Riemann integral, and will be the focus of this paper.

Riemann integral

Partition. Let [a,b] be a closed and bounded interval. Then a finite set of real numbers P = $\{x_o, x_1, x_2, \ldots\ldots x_n\}$ having the property that a = $x_o << x_1 << x_2 << \ldots\ldots << x_n = b$ is called the partition of [a,b]. For instance, P = $\{0, \frac{11}{44}, \frac{11}{22}, \frac{33}{44}, 1\}$ is called the partition of [0,1]. The partition P consists of n+1 points.

Let f be a bounded real valued function on [a,b] and let P = $\{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of [a,b].Let M_i, m_i be the bounds (supremum and infimum) of f in $[x_{i-1}, x_i]$.

Then the two sums

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

and L(P,f) = $\sum_{i=1}^{n} m_i \Delta x_i$ are respectively called the upper and the lower sums of f corresponding to the partition P.

If M, m are the bounds of f in [a,b], we have

$$m \leq m_i \leq M_i \leq M$$

$$m\sum_{i=1}^{n} \Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq M\sum_{i=1}^{n} \Delta x_i$$

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$,…………………………(1)

Now each partition gives rise to a pair of sums, the upper and the lower sums. By considering all partitions of [a,b], we get a set U of upper sums and a set L of lower sums. The inequality (1) shows that both these sets are bounded and so each set has the supremum and the infimum. The *infimum* of the set of upper sums is called the *upper integral* and the supremum of the set of lower sums is called the *lower integral* over [a,b].Thus,

$\int_a^{-b} f\,dx$ =inf{U(P,f): P∈∈ P[a,b]}

$\int_{-a}^{b} f dx$ = sup{L(P,f): P∈∈ P[a,b]}.

These two integrals may or may not be equal. When the two integrals are equal, we say that f is *Riemann integral* over [a,b] and the common value of these integrals is called the *Riemann integral* or simply integral of f over [a,b] and is written as $\int_a^b f$dx. For instance, a constant function is R-integrable function. The class of all the Riemann integrable functions over [a,b] is denoted by R[a,b].

Lebesgue Integral

A set A⊆[a,b] is said to be Lebesgue measurable set if its outer measure as well as inner measure are equal. The measure of A is denoted by mA. For instance, every open as well as closed sets are measurable.Let f be any bounded function on [a,b] and let P = {A_1,A_2,…..,A_n} be any partition of [a,b], where A_1,A_2,……,A_n are measurable subsets of [a,b] such that$\bigcup_{i=1}^{n} A_i$=[a,b] and m(A_i∩A_j)= 0, for i≠j. Such a partition of [a,b] would be called a measurable partition of [a,b]. We define

U(P,f) = $\sum_{i=1}^{n} (\sup_{x \in A_i} f$ (x)) mA_i and

L(P,f) = $\sum_{i=1}^{n} (\inf_{x \in A_i} f(x))$ mA_i

As the upper and lower Lebesgue sums of the function f corresponding to the partition P of [a,b]. Obviously U(P,f)≥L(P,f) for every partition P. The *infimum* of the set of all upper Lebesgue sums is called the *Upper Lebesgue integral* denoted as :

L $\int_a^{-b} f\ dx$ = inf U(P,f) ∀ Partitions P.

The *supremum* of the set of all lower Lebesgue sums is called the *Lower Lebesgue integral* denoted as :

L $\int_{-a}^{b} f\ dx$ = sup L(P,f) ∀ Partitions P.

A bounded function f on [a,b] is said to be *Lebesgue integrable* if L $\int_{-a}^{b} f\ dx$ = L $\int_a^{-b} f\ dx$,

and the common value of these two integrals is called the *Lebesgue integral* and is written as L$\int_a^b f\,dx$. Thus, we have L$\int_a^b f\,dx$ = L $\int_{-a}^{b} f\,dx$ = L $\int_a^{-b} f\,dx$.

The class of all the Lebesgue integrable functions is denoted by L[a,b]. Thus, if f is Lebesgue integrable, then we express by writing f∈L[a,b].

Lemma : Let f be a bounded function on [a,b]. Then for any two measurable partitions P_1 and P_2 of [a,b], we have

U(P_1,f) ≥ L(P_2,f) and L $\int_{-a}^{b} f\,dx \le$ L $\int_a^{-b} f\,dx$.

Remarks: Every upper (lower) Riemann integral is greater than(less than) or equal to every upper(lower) Lebesgue integral. i.e., R$\int_a^{-b} f\,dx \ge$ L $\int_a^{-b} f\,dx$, R$\int_{-a}^{b} f\,dx$ $\int_{-a}^{b} f\,dx \le$ L $\int_{-a}^{b} f\,dx$(Malik and Arora,*2010*).

Theorem. Every bounded Riemann integrable function over [a,b] is Lebesgue integrable. The converse need not be true.

Proof. If f Riemann integrable over [a,b], then

R$\int_{-a}^{b} f\,dx$ = R$\int_a^{-b} f\,dx$ = R$\int_a^b f$dx.

By the remarks, we have, R$\int_a^b f$dx ≤ L $\int_{-a}^{b} f\,dx$ ≤ L $\int_a^{-b} f\,dx$ ≤ R$\int_a^b f$dx

R$\int_a^b f$dx = L $\int_{-a}^{b} f\,dx$ = L $\int_a^{-b} f\,dx$ = L$\int_a^b f\,dx$.

Thus, every Riemann integrable function is Lebesgue integrable. The converse need not be true. It can be illustrated by the following example:

Let f be a function defined on the interval [0,1] as follows:

F(x) = 0, when x is rational

= 1, when x is irrational

This function is not Riemann integrable. For, Let P be a partition of [a,b]. Then U(P,f) = $\sum_{i=1}^{n} M_i\,\Delta x_i$ = 1.Δx_1+ 1.Δx_2+… … +1.Δx_n=1,and L(P,f) = $\sum_{i=1}^{n} m_i\,\Delta x_i = 0.\Delta x_1 + 0.\Delta x_2 + \cdots \ldots + 0.\Delta x_n$=0.

$\int_0^{-1} f\,dx$ = inf U(P,f) = 1

$\int_{-0}^{1} f\, dx$ = sup L(P,f) = 0.Therefore, $\int_{0}^{1} f\, dx$ does not exist.

For Lebesgue integrability, let A_1 be the set of all rational numbers and A_2 be the set of all irrational numbers in [0,1]. The partition P = $\{A_1,A_2\}$ is a measurable partition of [0,1] and $mA_1 = 0$, $mA_2 = 1$.

$L(P,f) = {}^{inf}_{A_1} f(x).mA_1 + {}^{inf}_{A_2} f(x).mA_2 = 0.mA_1 + 1.mA_2 = 1$

$U(P,f) = {}^{sup}_{A_1} f(x).mA_1 + {}^{sup}_{A_2} f(x).mA_2 = 0.mA_1 + 1.mA_2 = 1$

$${}^{sup}_{P} L(P,f) = {}^{inf}_{P} U\, {}^{inf}_{P} U(P,f)$$

$$L\int_{-0}^{1} f\, dx = L\int_{0}^{-1} f\, dx$$

f is Lebesgue integrable over [0,l] and $L\int_{0}^{1} f$ dx = 1.

Conclusion

It was found that every upper (lower) Riemann integral is greater than(less than) or equal to upper(lower) Lebesgue integral. The Lebesgue ingegral has been seen as generalization of the Riemann integral. It means that every Riemann integrable function is Lebesgue integrable. But the converse may or may not be true and was the focus of our paper.

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